YISHUN INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
Higher 2

CANDIDATE NAME

CG

INDEX NO

MATHEMATICS

9758/01

Paper 1

25 AUGUST 2021

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your CG, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

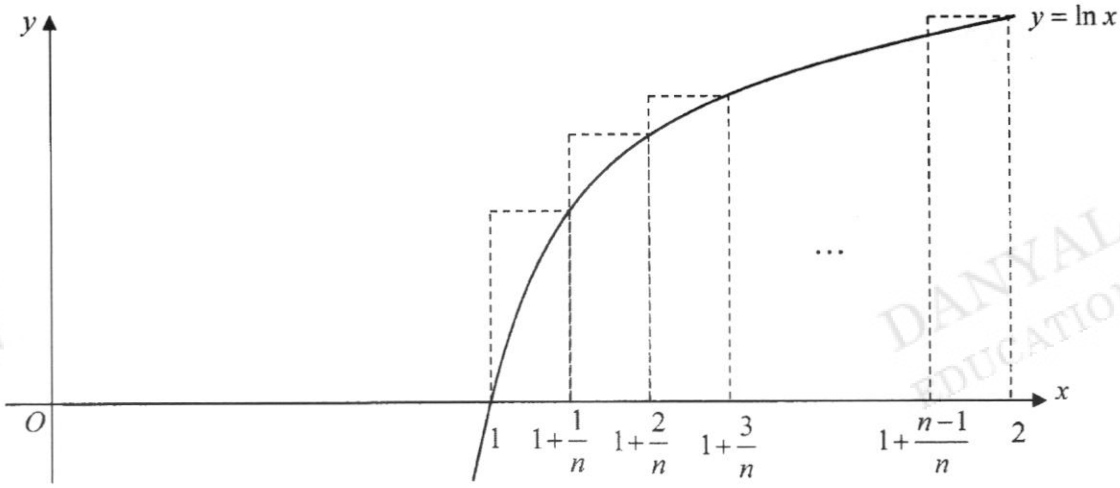
For Examiners' Use

Question	1	2	3	4	5	6	7
Marks							

Question	8	9	10	11	12	Total marks	/100
Marks							

This document consists of 22 printed pages and 2 blank pages.

1 **(a)** The diagram shows part of the graph of $y = \ln x$, with n rectangles of equal width, where n is a positive integer.



(i) Show that the total area of the n rectangles, A, is

$$A = \frac{1}{n}\sum_{r=1}^{n}\left[\ln(n+r)\right] - \ln n.$$ [2]

(ii) Evaluate $\lim_{n \to \infty} A$, giving your answer correct to 4 decimal places. [2]

(b) It is given that $f(x) = \frac{a}{x^2} + bx + c$, where a, b and c are constants. The curve with equation $y = f(x)$ has a minimum point with coordinates $(1, 2)$ and $\int_1^3 f(x)\, dx = \frac{20}{3}$. Find the equation of the curve. [4]

2 A curve has equation $y = f(x)$, where $f(x) = \begin{cases} \frac{1}{2}(x+5) & \text{for } x < -3, \\ 1 & \text{for } -3 \le x \le -1, \\ x^2 & \text{for } x > -1. \end{cases}$

(i) Sketch the curve for $-4 \le x \le 2$. [3]

(ii) On a separate diagram, sketch the curve with equation $y = f(2x-1)$, for $-2 \le x \le \frac{1}{2}$. [2]

3 Referred to the origin O, the points A and B have position vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. The point P with position vector $\mathbf{p}$ lies on AB such that $\mathbf{a}\cdot\mathbf{p} = \mathbf{b}\cdot\mathbf{p}$.

(i) Show that AB is perpendicular to OP. [2]

(ii) It is now given that $\mathbf{a}$ is perpendicular to $\mathbf{b}$. Show that

$$\mathbf{p} = \mathbf{a} + \frac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}(\mathbf{b} - \mathbf{a}).$$ [4]

4 Functions f and g are defined by

$$\mathrm{f} : x \mapsto 2 + \frac{3}{1 - x^2}, \qquad x \in \mathbb{R}, x < -1,$$

$$\mathrm{g} : x \mapsto -x^2 + 6x + a, \quad x \in \mathbb{R}, x \geq 0,$$

where a is a positive integer.

(i) Show that f has an inverse. [1]

(ii) Find $\mathrm{f}^{-1}(x)$ and state the domain of f^{-1}. [3]

(iii) State whether the composite function fg exists, justifying your answer. [3]

5 It is given that O is the origin and A is the point on the curve $y = xe^x$ where $x = 3$. The region bounded by the curve $y = xe^x$ and the line OA is rotated through 360° about the x-axis. Find the exact volume of the solid formed. [7]

6 **Do not use a calculator in answering this question.**

(i) One of the roots of the equation $2z^3 - 9z^2 + 30z + b = 0$ is $1 + a\mathrm{i}$, where a and b are non-zero real numbers. Find the values of a and b and the roots of this equation. [5]

(ii) Hence solve the equation $bz^3 + 30z^2 - 9z + 2 = 0$. [2]

7 (i) Sketch the curve with equation $y=\left|\frac{5-4x}{x+5}\right|$, stating the equations of the asymptotes. On the same diagram, sketch the line with equation $y=-2x-2$. [3]

(ii) Solve exactly the inequality $\left|\frac{5-4x}{x+5}\right|>-2x-2$. [4]

(iii) Hence, or otherwise, solve exactly the inequality $\left|\frac{10-4x}{x+10}\right|>-x-2$. [2]

8 It is given that $y=\sin\left(\ln(1+\mathrm{e}x)\right)$.

(i) Show that $(1+\mathrm{e}x)^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}(1+\mathrm{e}x)\frac{\mathrm{d}y}{\mathrm{d}x}=-\mathrm{e}^2y$. [2]

(ii) By further differentiation of the result in **(i)**, find the Maclaurin series for y, up to and including the term in x^3. [4]

(iii) By using appropriate standard series expansions from the List of Formulae (MF26), verify the correctness of the Maclaurin series for $y=\sin\left(\ln(1+\mathrm{e}x)\right)$ found in part **(ii)**. [2]

9 A curve C has equation $6x^2-4y^2=3xy^2$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{12x-3y^2}{8y+6xy}$. [2]

The points A and B on C each has x-coordinate 2. The tangents to C at A and B meet at the point M.

(ii) Find the exact coordinates of M. [5]

10 A curve C has parametric equations

$$x = a\cos^2\theta,$$
$$y = a\cos^2\theta\sin\theta,$$

for $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ and $a > 0$.

(i) Sketch C and state the Cartesian equation of the line of symmetry. [2]

(ii) Find the values of θ at the points where C meets the x-axis. [1]

(iii) Show that the area enclosed by the x-axis, and the part of C above the x-axis, is given by $\int_{\theta_1}^{\theta_2} 2a^2\cos^3\theta\sin^2\theta\,\mathrm{d}\theta$ where θ_1 and θ_2 should be stated. [3]

(iv) Hence, by expressing $\cos^3\theta$ as $\cos\theta\left(1-\sin^2\theta\right)$, find in terms of a, the exact total area enclosed by C. [3]

(v) It is given that the point $P\left(a\cos^2 p, a\cos^2 p\sin p\right)$ is on C. The point F is the midpoint of OP, where O is the origin. Find a Cartesian equation of the curve traced by F as p varies. [3]

11 A team of scientists is researching on the growth rate of a certain seaweed in an ocean. The length x metres, of the seaweed, at time t days (during a period of its growth) is proportional to the amount of water it contains. The seaweed absorbs water at a rate proportional to the length of the seaweed and loses water at a rate proportional to the square of the length of the seaweed. It is observed that the growth rates of the length of the seaweed are 0.49 metres per day and 0.96 metres per day when the lengths of the seaweed are 1 metre and 2 metres respectively.

(i) Show that x and t are related by the differential equation $\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1}{100}x(50-x)$. [3]

(ii) Given that the initial length of the seaweed is 0.5 metres, find an expression for x in terms of t. Hence find the time taken for the seaweed to reach a length of 45 metres. [7]

(iii) Sketch the graph of x against t. [2]

12 In order to train for the Open Water Swimming event in the University Sports Meet, Carol swims away from a shore towards the ocean. In her first minute of swimming, she covered a distance of 80 metres. Due to fatigue, the distance covered for each subsequent minute is 1% less than that in the previous minute.

However, there were ocean waves that pushed Carol back towards the shore. In her first minute of swimming, the waves pushed her back by a distance of 4 metres. As she swam further away from the shore, the waves got weaker and for each subsequent minute, she was pushed back by 0.05 metres less than that in the previous minute.

(i) Find the distance between Carol and the shore after the first 5 minutes. [3]

It is now given that the ocean waves became weaker away from the shore until they pushed Carol back towards the shore at 2 metres per minute from nth minute onwards.

(ii) Show that $n = 41$. [1]

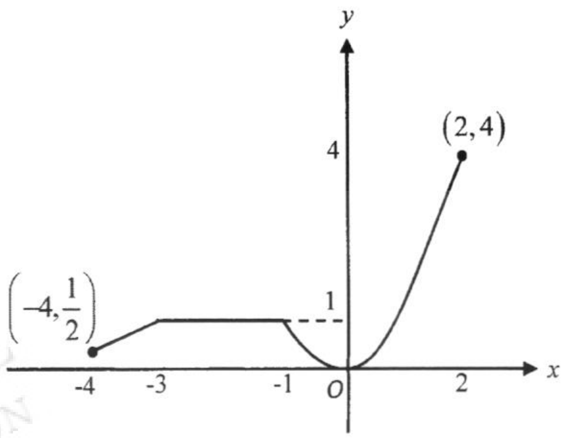
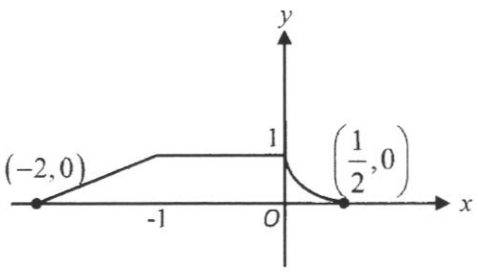
(iii) Hence find the time taken for the distance between Carol and the shore to be at least 5 kilometres, giving your answer correct to the nearest minute. [5]

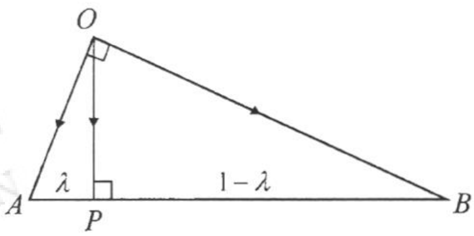
(iv) Find the greatest distance between Carol and the shore, and the time it took for her to reach this distance. Describe what happened to the distance Carol swam per minute after she had reached this greatest distance. [3]

Solutions 2021 JC2 Preliminary Examination Paper 1

1	Solution
(a) (i)	Total area of n rectangles, A $=\frac{1}{n}\ln\left(1+\frac{1}{n}\right)+\frac{1}{n}\ln\left(1+\frac{2}{n}\right)+...+\frac{1}{n}\ln\left(1+\frac{n}{n}\right)$ $=\frac{1}{n}\ln\left(\frac{n+1}{n}\right)+\frac{1}{n}\ln\left(\frac{n+2}{n}\right)+...+\frac{1}{n}\ln\left(\frac{2n}{n}\right)$ $=\frac{1}{n}\{[\ln(n+1)-\ln n]+[\ln(n+2)-\ln n]+...+[\ln(2n)-\ln n]\}$ $=\frac{1}{n}[\ln(n+1)+\ln(n+2)+...+\ln(2n)]-\frac{1}{n}(n\ln n)$ $=\frac{1}{n}\sum_{r=1}^{n}[\ln(n+r)]-\ln n$ (shown)
(ii)	$\lim_{n\to\infty} A=\int_1^2 \ln x \, dx$ $= 0.3863$ (4 d.p.)

1	Solution
(b)	At $(1,2)$, $a+b+c=2$ ------(1) $f'(x)=\frac{-2a}{x^3}+b$ $-2a+b=0$ ------(2) $\int_1^3 f(x)\,dx=\left[-\frac{a}{x}+b\frac{x^2}{2}+cx+d\right]_1^3$ $\left[-\frac{a}{3}+b\frac{9}{2}+3c+d\right]-\left[-a+b\frac{1}{2}+c+d\right]=\frac{20}{3}$ $\frac{2}{3}a+4b+2c=\frac{20}{3}$ ------(3) From GC $a=1, b=2, c=-1$ $y=\frac{1}{x^2}+2x-1$

2	Solution
(i)	
(ii)	

3	Solution
(i)	$\mathbf{a}\bullet\mathbf{p} = \mathbf{b}\bullet\mathbf{p}$ $\mathbf{b}\bullet\mathbf{p} - \mathbf{a}\bullet\mathbf{p} = 0$ $(\mathbf{b} - \mathbf{a})\bullet\mathbf{p} = 0$ $\overrightarrow{AB}\bullet\overrightarrow{OP} = 0$ Hence, AB is perpendicular to OP.
(ii)	 **<u>Method 1</u>** Let $AP:PB = \lambda:1-\lambda$ By ratio theorem, $\mathbf{p} = \lambda\mathbf{b} + (1-\lambda)\mathbf{a}$

$\mathbf{a}\bullet\mathbf{p} = \mathbf{b}\bullet\mathbf{p}$

$\mathbf{a}\bullet(\lambda\mathbf{b} + (1-\lambda)\mathbf{a}) = \mathbf{b}\bullet(\lambda\mathbf{b} + (1-\lambda)\mathbf{a})$

$\lambda\mathbf{a}\bullet\mathbf{b} + (1-\lambda)\mathbf{a}\bullet\mathbf{a} = \lambda\mathbf{b}\bullet\mathbf{b} + (1-\lambda)\mathbf{b}\bullet\mathbf{a}$

$|\mathbf{a}|^2 - \lambda|\mathbf{a}|^2 = \lambda|\mathbf{b}|^2 \qquad (\because \mathbf{a}\bullet\mathbf{a} = |\mathbf{a}|^2,\ \mathbf{b}\bullet\mathbf{b} = |\mathbf{b}|^2,\ \mathbf{a}\bullet\mathbf{b} = 0)$

$$\lambda = \frac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}$$

Hence, $\mathbf{p} = \dfrac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}\mathbf{b} + \left(1 - \dfrac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}\right)\mathbf{a}$

$$= \mathbf{a} + \frac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}(\mathbf{b} - \mathbf{a})$$

Method 2

The projection vector of $\overrightarrow{AO}$ onto $\overrightarrow{AB}$ is

$$\overrightarrow{AP} = \left(\frac{\overrightarrow{AO}\bullet\overrightarrow{AB}}{|\overrightarrow{AB}|}\right)\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$$

$$= \left(\frac{-\mathbf{a}\bullet(\mathbf{b}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|}\right)\frac{(\mathbf{b}-\mathbf{a})}{|\mathbf{b}-\mathbf{a}|}$$

$$= \left(\frac{-\mathbf{a}\bullet\mathbf{b} + \mathbf{a}\bullet\mathbf{a}}{(\mathbf{b}-\mathbf{a})\bullet(\mathbf{b}-\mathbf{a})}\right)(\mathbf{b}-\mathbf{a})$$

$$= \left(\frac{-\mathbf{a}\bullet\mathbf{b} + |\mathbf{a}|^2}{|\mathbf{b}|^2 - 2\mathbf{a}\bullet\mathbf{b} + |\mathbf{a}|^2}\right)(\mathbf{b}-\mathbf{a}) \quad (\because \mathbf{a}\bullet\mathbf{a} = |\mathbf{a}|^2,\ \mathbf{b}\bullet\mathbf{b} = |\mathbf{b}|^2)$$

$$= \frac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}(\mathbf{b}-\mathbf{a}) \qquad (\because \mathbf{a}\bullet\mathbf{b} = 0)$$

Hence, $\mathbf{p} = \overrightarrow{OA} + \overrightarrow{AP}$

$$= \mathbf{a} + \frac{|\mathbf{a}|^2}{|\mathbf{a}|^2 + |\mathbf{b}|^2}(\mathbf{b}-\mathbf{a})$$

Method 3

Note that OAB and PAO are similar triangles,

$$\therefore \frac{|\mathbf{a}|}{AP} = \frac{\sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2}}{|\mathbf{a}|}$$

$$AP = \frac{|\mathbf{a}|^2}{\sqrt{|\mathbf{a}|^2 + |\mathbf{b}|^2}}$$

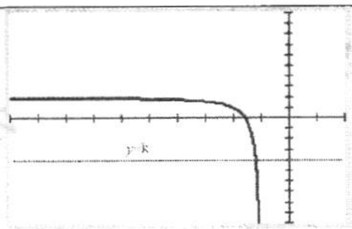
$$\overrightarrow{AP} = AP\left(\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}\right)$$

$$\mathbf{p}-\mathbf{a}=\frac{|\mathbf{a}|^2}{\sqrt{|\mathbf{a}|^2+|\mathbf{b}|^2}}\left(\frac{\mathbf{b}-\mathbf{a}}{\sqrt{|\mathbf{a}|^2+|\mathbf{b}|^2}}\right)$$

$$\mathbf{p}=\mathbf{a}+\frac{|\mathbf{a}|^2}{|\mathbf{a}|^2+|\mathbf{b}|^2}(\mathbf{b}-\mathbf{a})$$

4	Solution

(i)



Any horizontal line $y=k$, where $k\in\mathbb{R}$, cuts the graph of f **at most once**, therefore f is one-one and f has an inverse.

(ii)

Let $y=2+\frac{3}{1-x^2}$

$\frac{3}{1-x^2}=y-2$

$1-x^2=\frac{3}{y-2}$

$x^2=1-\frac{3}{y-2}$

$x=\pm\sqrt{1-\frac{3}{y-2}}$

$x=\sqrt{1-\frac{3}{y-2}}$ or $x=-\sqrt{1-\frac{3}{y-2}}$

(rej. since $x<-1$)

$\therefore \mathrm{f}^{-1}(x)=-\sqrt{1-\frac{3}{x-2}}$, $x<2$

(iii)

$\mathrm{g}(x)=-x^2+6x+a$

$\mathrm{g}(x)=-(x-3)^2+a+9$

$D_{\mathrm{f}}=(-\infty,-1)$, $R_{\mathrm{g}}=(-\infty,a+9]$

Since $a+9>-1$ for $a>0$, $R_{\mathrm{g}}\not\subseteq D_{\mathrm{f}}$

Therefore, fg does not exist.

5	**Solution**

Volume generated

$$=\frac{1}{3}\pi\left(3e^{3}\right)^{2}(3)-\pi\int_{0}^{3}x^{2}e^{2x}dx=9\pi e^{6}-\pi\left\{\left[\frac{1}{2}x^{2}e^{2x}\right]_{0}^{3}-\int_{0}^{3}xe^{2x}dx\right\}$$

$u=x^{2}$, $\frac{dv}{dx}=e^{2x}$

$\frac{du}{dx}=2x$, $v=\frac{1}{2}e^{2x}$

$$=9\pi e^{6}-\pi\left\{\frac{9}{2}e^{6}-\left[\frac{x}{2}e^{2x}\right]_{0}^{3}+\int_{0}^{3}\frac{1}{2}e^{2x}dx\right\}$$

$u=x$, $\frac{dv}{dx}=e^{2x}$

$\frac{du}{dx}=1$, $v=\frac{1}{2}e^{2x}$

$$=9\pi e^{6}-\pi\left(\frac{9}{2}e^{6}-\frac{3}{2}e^{6}+\left[\frac{e^{2x}}{4}\right]_{0}^{3}\right)$$

$$=9\pi e^{6}-\pi\left(3e^{6}+\frac{e^{6}}{4}-\frac{1}{4}\right)$$

$$=9\pi e^{6}-\frac{\pi}{4}(13e^{6}-1)$$

$$=\frac{\pi}{4}(23e^{6}+1)$$

6	**Solution**

(i) Since the coefficients of the equation are real, $1-a\text{i}$ is also a root of the equation.

Quadratic factor

$$=\left(z-(1+a\text{i})\right)\left(z-(1-a\text{i})\right)$$

$$=\left(z-1-a\text{i}\right)\left(z-1+a\text{i}\right)$$

$$=\left(z-1\right)^{2}-\left(a\text{i}\right)^{2}$$

$$=z^{2}-2z+1+a^{2}$$

By comparison,

$$2z^{3}-9z^{2}+30z+b=\left(z^{2}-2z+1+a^{2}\right)\left(cz+d\right)$$

Comparing coefficient of $z^{3}\Rightarrow c=2$

Comparing coefficient of $z^{2}\Rightarrow -9=d-2(2)\Rightarrow d=-5$

Comparing coefficient of $z \Rightarrow 30=-2(-5)+\left(1+a^{2}\right)(2)$

$\Rightarrow a^{2}=9$

$\Rightarrow a=3$ or $a=-3$

Comparing constant $\Rightarrow b=-5(1+9)=-50$

$$\therefore\ 2z^{3}-9z^{2}+30z-50=\left(z^{2}-2z+10\right)\left(2z-5\right)=0$$

The roots are $1+3\text{i}$, $1-3\text{i}$ and $\frac{5}{2}$.

Alternative Method (Not recommended)

$(1+ai)^3$

$=1+3(ai)+3(ai)^2+(ai)^3$

$=1+3ai-3a^2-a^3\mathrm{i}$

$=\left(1-3a^2\right)+\mathrm{i}\left(3a-a^3\right)$

Since $1+ai$ is a root of the equation,

$$2(1+ai)^3-9(1+ai)^2+30(1+ai)+b=0$$

$$2\left[\left(1-3a^2\right)+\mathrm{i}\left(3a-a^3\right)\right]-9\left(1+2ai-a^2\right)+30(1+ai)+b=0$$

$$\left(23+b+3a^2\right)+\mathrm{i}\left(18a-2a^3\right)=0$$

Comparing the real part: $23+b+3a^2=0$ --- (1)
Comparing the imaginary part: $18a-2a^3=0$ --- (2)

From (2): $a=0$ (rej $\because a\neq 0$) or $a=3$ or $a=-3$

From (1): $23+b+3(-3)^2=0 \Rightarrow b=-50$

Since the coefficients of the equation are real, $1-3\mathrm{i}$ and $1+3\mathrm{i}$ are roots of the equation.

Quadratic factor

$=(z-(1+3\mathrm{i}))(z-(1-3\mathrm{i}))$

$=(z-1-3\mathrm{i})(z-1+3\mathrm{i})$

$=(z-1)^2-(3\mathrm{i})^2$

$=z^2-2z+10$

By comparison,

$2z^3-9z+30z-50=\left(z^2-2z+10\right)(cz+d)$

Comparing coefficient of $z^3 \Rightarrow c=2$
Comparing constant $\Rightarrow 10d=-50 \quad \Rightarrow d=-5$

$\therefore\ 2z^3-9z^2+30z-50=\left(z^2-2z+10\right)(2z-5)=0$

The roots are $1+3\mathrm{i}$, $1-3\mathrm{i}$ and $\dfrac{5}{2}$.

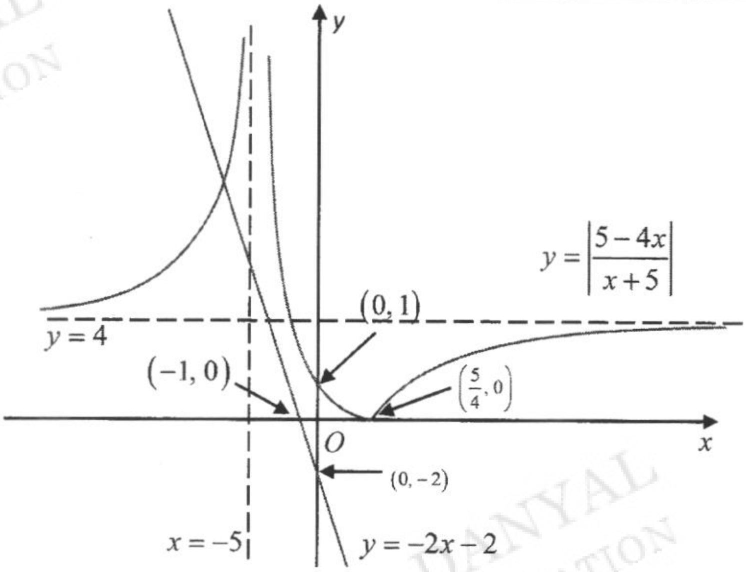
(ii) $bz^3+30z^2-9z+2=0$

Divide throughout by z^3,

$b+\dfrac{30}{z}-\dfrac{9}{z^2}+\dfrac{2}{z^3}=0$

$2\left(\dfrac{1}{z}\right)^3-9\left(\dfrac{1}{z}\right)^2+30\left(\dfrac{1}{z}\right)+b=0$

	Replace z with $\frac{1}{z}$, $\frac{1}{z}=\frac{5}{2}$ or $\frac{1}{z}=1+3\text{i}$ or $\frac{1}{z}=1-3\text{i}$ $z=\frac{2}{5}$ or $z=\frac{1}{1+3\text{i}}=\frac{1}{1+3\text{i}}\times\frac{1-3\text{i}}{1-3\text{i}}=\frac{1}{10}-\frac{3}{10}\text{i}$ or $z=\frac{1}{1-3\text{i}}\times\frac{1+3\text{i}}{1+3\text{i}}=\frac{1}{10}+\frac{3}{10}\text{i}$

7	**Solution**
(i)	
(ii)	From the graph, the point of intersection occurs when $x<-5$. $x<-5 \Rightarrow 5-4x>0,\ x+5<0$ $\therefore \frac{5-4x}{x+5}<0$ and $\left\|\frac{5-4x}{x+5}\right\|=-\frac{5-4x}{x+5}$ $-\frac{5-4x}{x+5}=-2x-2$ $-5+4x=(-2x-2)(x+5)$ $-5+4x=-2x^2-12x-10$ $2x^2+16x+5=0$ $x=\frac{-16\pm\sqrt{16^2-4(2)(5)}}{2(2)}$ $=\frac{-16\pm\sqrt{216}}{4}$ $=\frac{-16\pm 6\sqrt{6}}{4}$ $=-4\pm\frac{3}{2}\sqrt{6}$ Since $x<-5$, hence $x=-4-\frac{3}{2}\sqrt{6}$. $-4-\frac{3}{2}\sqrt{6}<x<-5$ or $x>-5$

	Alternative Method (strongly discouraged):
	$\left(\frac{5-4x}{x+5}\right)^2=(-2x-2)^2$
	$(5-4x)^2=(-2x-2)^2(x+5)^2$
	$(5-4x)^2=\left[(-2x-2)(x+5)\right]^2$
	$(5-4x)^2-\left[(-2x-2)(x+5)\right]^2=0$
	Since $a^2-b^2=(a-b)(a+b)$,
	$\left[(5-4x)-(-2x-2)(x+5)\right]\left[(5-4x)+(-2x-2)(x+5)\right]=0$
	$\left[(5-4x)-\left(-2x^2-12x-10\right)\right]\left[(5-4x)+\left(-2x^2-12x-10\right)\right]=0$
	$\left[2x^2+8x+15\right]\left[-2x^2-16x-5\right]=0$
	$2x^2+8x+15=0$ or $-2x^2-16x-5=0$
	For $2x^2+8x+15=0=0$
	$x=\frac{-8\pm\sqrt{8^2-4(2)(15)}}{2(2)}$
	$=\frac{-8\pm\sqrt{-56}}{4}$
	There are no real roots.
	For $-2x^2-16x-5=0$
	$x=\frac{16\pm\sqrt{16^2-4(-2)(-5)}}{2(-2)}$
	$=\frac{16\pm\sqrt{216}}{-4}$
	$=\frac{16\pm 6\sqrt{6}}{-4}$
	$=-4\pm\frac{3}{2}\sqrt{6}$
	Since $x<-5$, hence $x=-4-\frac{3}{2}\sqrt{6}$.
	$-4-\frac{3}{2}\sqrt{6}<x<-5$ or $x>-5$
(iii)	$\left\lvert\frac{10-4x}{x+10}\right\rvert>-x-2$
	$\left\lvert\frac{2(5-2x)}{2(0.5x+5)}\right\rvert>-x-2$

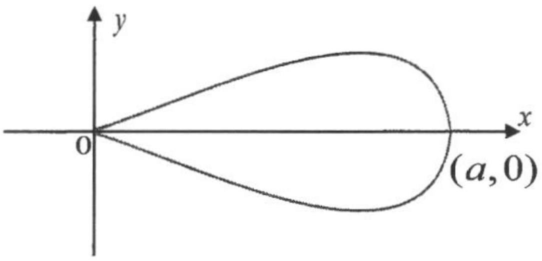
	$\left\|\dfrac{5-4\left(\frac{x}{2}\right)}{\left(\frac{x}{2}\right)+5}\right\| > -2\left(\frac{x}{2}\right)-2$ Hence replace x with $\frac{x}{2}$, $-4-\frac{3}{2}\sqrt{6} < \frac{x}{2} < -5$ or $\frac{x}{2} > -5$ $-8-3\sqrt{6} < x < -10$ or $x > -10$

8	Solution
(i)	$y = \sin(\ln(1+ex))$ $\frac{dy}{dx} = \frac{e}{1+ex}\cos(\ln(1+ex))$ $(1+ex)\frac{dy}{dx} = e\cos(\ln(1+ex))$ $(1+ex)\frac{d^2y}{dx^2} + e\frac{dy}{dx} = e(-\sin(\ln(1+ex)))\frac{e}{(1+ex)}$ $(1+ex)^2\frac{d^2y}{dx^2} + e(1+ex)\frac{dy}{dx} = -e^2y$ (Shown)
(ii)	$(1+ex)^2\frac{d^2y}{dx^2} + e(1+ex)\frac{dy}{dx} = -e^2y$ $(1+ex)^2\frac{d^3y}{dx^3} + 2e(1+ex)\frac{d^2y}{dx^2} + e(1+ex)\frac{d^2y}{dx^2} + e^2\frac{dy}{dx} = -e^2\frac{dy}{dx}$ $(1+ex)^2\frac{d^3y}{dx^3} + 3e(1+ex)\frac{d^2y}{dx^2} + 2e^2\frac{dy}{dx} = 0$ When $x=0, y=0, \frac{dy}{dx} = e, \frac{d^2y}{dx^2} = -e^2, \frac{d^3y}{dx^3} = e^3$ $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$ $y = ex - \frac{e^2}{2}x^2 + \frac{e^3}{6}x^3 + \ldots$
(iii)	$y = \sin\left(ex - \frac{(ex)^2}{2} + \frac{(ex)^3}{3} + \ldots\right)$ $= \left(ex - \frac{e^2x^2}{2} + \frac{e^3x^3}{3} + \ldots\right) - \frac{\left(ex - \frac{e^2x^2}{2} + \frac{e^3x^3}{3} + \ldots\right)^3}{6} + \ldots$

	$=ex-\frac{e^2x^2}{2}+\frac{e^3x^3}{3}-\frac{e^3x^3}{6}+...$ $=ex-\frac{e^2x^2}{2}+\frac{e^3x^3}{6}+...$

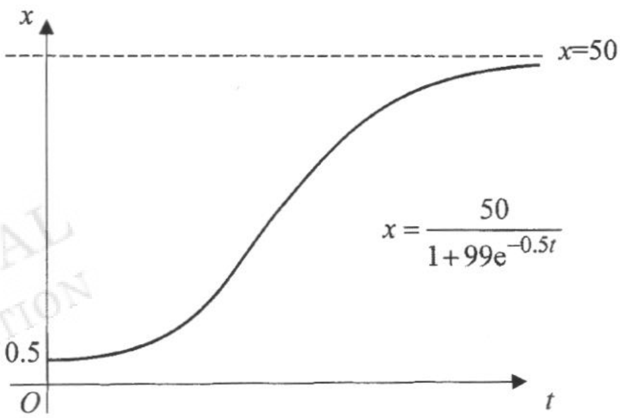
9	Solution
(i)	$6x^2-4y^2=3xy^2$ $12x-8y\frac{dy}{dx}=3y^2+6xy\frac{dy}{dx}$ $\frac{dy}{dx}(8y+6xy)=12x-3y^2$ $\frac{dy}{dx}=\frac{12x-3y^2}{8y+6xy}$
(ii)	When $x=2$, $6x^2-4y^2-3xy^2=0$ $24-4y^2-6y^2=0$ $10y^2=24$ $y=\pm\sqrt{\frac{12}{5}}$ At $\left(2,\sqrt{\frac{12}{5}}\right)$ $\frac{dy}{dx}=\frac{24-3\left(\frac{12}{5}\right)}{20\left(\sqrt{\frac{12}{5}}\right)}=\frac{21}{25}\sqrt{\frac{5}{12}}$ $y-\sqrt{\frac{12}{5}}=\frac{21}{25}\sqrt{\frac{5}{12}}(x-2)---(1)$ At $\left(2,-\sqrt{\frac{12}{5}}\right)$ $\frac{dy}{dx}=\frac{24-3\left(\frac{12}{5}\right)}{20\left(-\sqrt{\frac{12}{5}}\right)}=-\frac{21}{25}\sqrt{\frac{5}{12}}$ $y+\sqrt{\frac{12}{5}}=-\frac{21}{25}\sqrt{\frac{5}{12}}(x-2)---(2)$ (1)+ (2): $y=0$

	$\sqrt{\frac{12}{5}}=-\frac{21}{25}\sqrt{\frac{5}{12}}(x-2)$ $x-2=\sqrt{\frac{12}{5}}\left(-\frac{25}{21}\sqrt{\frac{12}{5}}\right)$ $x=-\frac{20}{7}+2=-\frac{6}{7}$ Coordinates are $\left(-\frac{6}{7},0\right)$

10	Solution
(i)	 Line of symmetry: $y=0$
(ii)	GC: When $y=0$, $a\cos^2\theta\sin\theta=0$ $\cos^2\theta=0$ or $\sin\theta=0$ $\theta=\frac{\pi}{2}$ or $-\frac{\pi}{2}$ $\theta=0$ $\therefore\theta=0$ or $\frac{\pi}{2}$ or $-\frac{\pi}{2}$
(iii)	$x=a\cos^2\theta$ $\frac{dx}{d\theta}=-2a\cos\theta\sin\theta$ When $x=0,\theta=\frac{\pi}{2}$ When $x=a,\theta=0$ Area $=\int_0^a y\,dx$ $=\int_{\frac{\pi}{2}}^{0}\left(a\cos^2\theta\sin\theta\right)\left(-2a\cos\theta\sin\theta\right)d\theta$ $=\int_0^{\frac{\pi}{2}}2a^2\cos^3\theta\sin^2\theta\,d\theta$ (Shown)
(iv)	Total area enclosed by C $=2\int_0^a y\,dx$

	$= 4a^2\int_0^{\frac{\pi}{2}} \cos^3\theta \sin^2\theta \,\mathrm{d}\theta$ $= 4a^2\int_0^{\frac{\pi}{2}} \cos\theta(1-\sin^2\theta)(\sin^2\theta)\,\mathrm{d}\theta$ $= 4a^2\int_0^{\frac{\pi}{2}} \cos\theta(\sin^2\theta-\sin^4\theta)\,\mathrm{d}\theta$ $= 4a^2\int_0^{\frac{\pi}{2}} \cos\theta\sin^2\theta-\cos\theta\sin^4\theta\,\mathrm{d}\theta$ $= 4a^2\left[\frac{\sin^3\theta}{3}-\frac{\sin^5\theta}{5}\right]_0^{\frac{\pi}{2}}$ $= 4a^2\left[\frac{1}{3}-\frac{1}{5}\right] = 4a^2\left(\frac{2}{15}\right)$ $= \frac{8}{15}a^2$
(v)	Midpoint of $OP = \left(\frac{a\cos^2 p}{2}, \frac{a\cos^2 p\sin p}{2}\right)$ $x = \frac{a\cos^2 p}{2},\ y = \frac{a\cos^2 p\sin p}{2}$ $y = \frac{a\cos^2 p\sin p}{2} = x\sin p$ $\therefore \sin p = \frac{y}{x}$ $x = \frac{a\cos^2 p}{2} = \frac{a(1-\sin^2 p)}{2} = \frac{a}{2}\left(1-\frac{y^2}{x^2}\right)$ $\frac{2x}{a} = \left(1-\frac{y^2}{x^2}\right)$ $2x^3 = a\left(x^2-y^2\right)$

11	Solution
(i)	$\frac{\mathrm{d}x}{\mathrm{d}t} = ax - bx^2$ $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.49, \qquad x = 1$ $0.49 = a - b ----(1)$ $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.96, \qquad x = 2$ $0.96 = 2a - 4b ----(2)$ Solving, $a = 0.5,\ b = 0.01$ $\frac{\mathrm{d}x}{\mathrm{d}t} = 0.5x - 0.01x^2$

	$\frac{dx}{dt}=\frac{1}{100}x(50-x)$
(ii)	$\int\frac{1}{x(50-x)}\ dx=\int 0.01\ dt$ $\frac{1}{50}\int\frac{1}{50-x}+\frac{1}{x}\ dx=\int 0.01\ dt$ $\int\frac{1}{50-x}+\frac{1}{x}\ dx=\int 0.5\ dt$ $\ln\left\|\frac{x}{50-x}\right\|=0.5t+C$ $\frac{x}{50-x}=\pm e^{0.5t+C}=Ae^{0.5t}$ When $t=0,\ x=0.5$ $A=\frac{1}{99}$ $\frac{x}{50-x}=\frac{1}{99}e^{0.5t}$ $\frac{50-x}{x}=99e^{-0.5t}$ $\frac{50}{x}=1+99e^{-0.5t}$ $x=\frac{50}{1+99e^{-0.5t}}$ $45=\frac{50}{1+99e^{-0.5t}}$ From GC, it takes 13.6 days to reach 45 m.
(iii)	

12	Solution

(i)

Total distance swam by Carol in 5 minutes

$=\dfrac{80(1-0.99^5)}{1-0.99}$

$=392.08$

Total distance pushed back by the waves in 5 minutes

$=\dfrac{5}{2}[2(4)+(5-1)(-0.05)]$

$=19.5$

Distance between Carol and the shore after the first 5 minutes

$=392.08-19.5$

$=372.58=373$ m (3 s.f.)

(ii)

$U_n=2$

$4+(n-1)(-0.05)=2$

$-0.05(n-1)=-2$

$n-1=40$

$n=41$

(iii)

Distance between Carol and the shore in the first 40 minutes

$=\dfrac{80(1-0.99^{40})}{1-0.99}-\dfrac{40}{2}[2(4)+(40-1)(-0.05)]$

$=2527.2$

Remaining distance from 5 km $=5000-2527.2$

$=2472.8$

Let t be the number of minutes after the 40th minute.

$\dfrac{80\left(0.99^{40}\right)\left(1-0.99^t\right)}{1-0.99}-2t\geq 2472.8$

From GC, $66.387\leq t\leq 1429.5$

$\therefore t\geq 67$

Therefore, minimum time (in nearest minute) is

40 + 67 = <u>107 minutes</u>

(iv)

Consider $d=\dfrac{80\left(0.99^{40}\right)\left(1-0.99^t\right)}{1-0.99}-2t+2527.2$

GC:

t	Distance, d
327	7024.9187
328	7024.9196 (largest)
329	7024.9004

$328+40=368$

Greatest distance is 7020 (3s.f.) metres, occurs at 368 minutes.

The distance Carol swam per minute drops below 2 metres per minute.

YISHUN INNOVA JUNIOR COLLEGE
JC 2 PRELIMINARY EXAMINATION
Higher 2

CANDIDATE NAME	

CG		INDEX NO	

MATHEMATICS

9758/02

Paper 2

15 SEPTEMBER 2021

3 hours

Candidates answer on the Question Paper.
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your CG, index number and name on the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the Question Paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For Examiners' Use

Question	1	2	3	4	5	6
Marks						

Question	7	8	9	10	**Total marks**
Marks					/100

This document consists of 22 printed pages and 2 blank pages.

Section A: Pure Mathematics [40 marks]

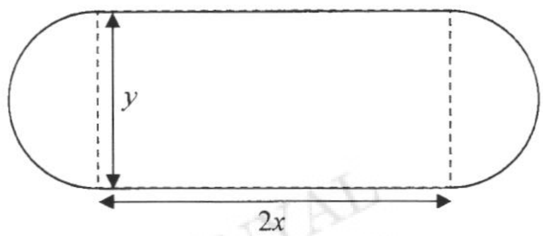
1 **(i)** Prove by the method of differences that $\sum_{r=3}^{n}\frac{1}{r^2-2r}=\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n-1}-\frac{1}{n}\right)$. [4]

(ii) Explain why $\sum_{r=3}^{\infty}\frac{1}{r^2-2r}$ is a convergent series, and state the value of the sum to infinity. [2]

(iii) Hence show that $\sum_{r=4}^{\infty}\frac{1}{(r-1)^2}<\frac{5}{12}$. [2]

2 [It is given that a sphere of radius r has surface area $4\pi r^2$ and volume $\frac{4}{3}\pi r^3$.]

A capsule is to be constructed using the curved surface of a cylinder and the curved surface of a hemisphere at each end. The length of the cylinder is $2x$ metres and the diameter of each hemisphere is y metres. The cross-sectional view of the capsule is shown in the diagram below.



It is given that the construction cost per square metre for the cylinder is \$$3k$ and that for the hemispheres is \$$5k$, where k is a constant.

Given that the volume of the capsule is π m^3, find, using differentiation, the exact values of x and y that give a minimum cost for its construction. [9]

3 **Do not use a calculator in answering this question.**

(a) The complex number z is given by $z=\frac{(1-\mathrm{i})^3}{\sqrt{2}(a+\mathrm{i})^2}$, where $a<0$.

(i) Given that $|z|=\frac{1}{2}$, show that $\arg z=-\frac{5\pi}{12}$. [5]

(ii) Hence find the smallest positive integer n for which z^n has equal real and imaginary parts. [2]

(b) The complex number q is given by $\frac{\mathrm{e}^{-\mathrm{i}\theta}}{\mathrm{e}^{\mathrm{i}\theta}-\mathrm{i}}$, where $0<\theta<\frac{\pi}{2}$. Show that

$$\mathrm{Re}(q)=\frac{1}{2}(1+2\sin\theta).$$ [4]

4 The line l has equation $\frac{x+1}{2}=\frac{y-a}{b}=\frac{z-4}{3}$. The plane p has equation $x+y-z-11=0$.

(i) If l and p do not intersect, what can be said about the values of a and b? [3]

For the rest of the question, use $a=-2$ and $b=1$.

(ii) Find the coordinates of the point B, the foot of perpendicular from the point $A(-1, a, 4)$ to p. [2]

(iii) Hence, or otherwise, find a vector equation of the line of reflection of l in p. [2]

(iv) Find the possible position vectors of the point C on l which is a distance of $\sqrt{164}$ from B. [3]

(v) Find the angle BCA. [2]

Section B: Probability and Statistics [60 marks]

5 12 students consisting of 4 from Class A, 5 from Class B and 3 from Class C participate in a team bonding activity where they are divided into 3 groups of 4.

(i) Find the number of ways such that each group has at most two students from Class A. [3]

During lunchtime, all the students from Class A and Class B are seated at a round table with 10 chairs.

(ii) How many different seating arrangements can be formed if no two students from Class A are seated next to each other? [2]

6 Events A, B and C are such that $\text{P}(A)=0.5$, $\text{P}(B)=0.6$ and $\text{P}(C)=0.65$.
It is given that $\text{P}(A|B)=0.45$, $\text{P}(B\cap C)=0.4$ and $\text{P}(A\cap B\cap C)=0.1$.

(i) Find $\text{P}(A\cap B\cap C')$. [2]

(ii) It is also given that events A and C are independent. Find $\text{P}(A\cup C')$. [2]

(iii) Given instead that events A and C are **not** independent, find the greatest and least possible values of $\text{P}(A'\cap B'\cap C')$. [4]

7 A game is played with a set of 9 cards. Two of the cards are numbered 1, three of the cards are numbered 2 and four of the cards are numbered 3. A player randomly draws three cards without replacement. The random variable X denotes the sum of the numbers on the three cards drawn.

(i) Find the probability distribution of X. [4]

The player wins \$$w$ if the sum of the numbers on the three cards drawn is an odd number, and loses \$$w$ otherwise. The random variable Y denotes the player's profit in dollars.

(ii) Given that $\text{E}(Y)=-0.8$, find $\text{Var}(Y)$. [4]

(iii) Find the probability that the player wins \$$2w$ at the end of 10 rounds. [2]

8 In order to get a vaccination, people will go to either polyclinics or community clubs. The waiting time for vaccination in polyclinics follows a normal distribution with mean 25 minutes and standard deviation 6 minutes. The waiting time for vaccination in community clubs follows a normal distribution with mean 20 minutes and standard deviation 3 minutes.

(i) Two people waiting for their vaccinations in polyclinics are chosen at random. Find the probability that the waiting time for one of them is more than 30 minutes and the waiting time for the other is less than 30 minutes. [3]

(ii) Jim is waiting for his vaccination in a polyclinic and Tina is waiting for her vaccination in a community club. Find the probability that the difference between their waiting times is at least 3 minutes. [4]

A random sample of 5 people waiting for their vaccinations in polyclinics and 15 people waiting for their vaccinations in community clubs is taken.

(iii) Find the probability that their average waiting time is less than 21 minutes. State an assumption needed for your calculations to be valid. [5]

9 A chocolate manufacturer claims that the mean number of calories in a packet of chocolate is 300. A random sample of 50 packets of chocolate is selected and the number of calories, x, in each packet is measured. The results are summarised by:

$$\sum(x-300)=36, \quad \sum(x-300)^2=612.$$

(i) Find unbiased estimates of the population mean and variance. [2]

A nutritionist believes that the chocolate manufacturer has understated the mean number of calories in a packet of chocolate.

(ii) Test, at the 5% significance level, whether the nutritionist's belief is correct. [4]

(iii) Explain why there is no need for the nutritionist to know anything about the population distribution of the number of calories in the packets of chocolate. [1]

(iv) Explain, in the context of the question, the meaning of "at the 5% significance level". [1]

After receiving feedback from customers, the chocolate manufacturer improves the quality of the ingredients used. A random sample of 40 packets of the improved chocolate is taken. The mean number of calories of the 40 packets of chocolate is k and the variance is 10.

(v) Find the range of values of k such that there is sufficient evidence to conclude that the mean number of calories in a packet of the improved chocolate is not 300 at the 3% significance level. [4]

10 A factory produces a large number of bottles. Based on past records, 4% of the bottles are defective.

A departmental store manager wishes to purchase bottles from the factory. To decide whether to accept or reject a batch of bottles, the manager designs a sampling process. He takes a random sample of 30 bottles. The batch is accepted if there are no defective bottles and is rejected if there are more than 2 defective bottles. Otherwise, a second random sample of 30 bottles is taken. The batch is then accepted if there are fewer defective bottles in the second sample and is rejected otherwise.

(i) Find the probability of accepting a batch of bottles. [2]

(ii) If a batch is accepted, find the probability that there are at most 2 defective bottles found in the sampling process. [3]

Another departmental store manager purchases n randomly chosen boxes of 30 bottles each.

(iii) Find the probability that a box has exactly 20 fewer defective bottles than non-defective bottles. [2]

(iv) Find the greatest value of n if the probability that there are at least 2 boxes with exactly 20 fewer defective bottles than non-defective bottles is at most 0.05. [3]

(v) Given that $n = 50$, find the probability that the mean number of defective bottles in a box is less than 1. [3]

2021 JC2 H2MA Preliminary Examination Paper 2 Solution

1 (i)	$\frac{1}{r^2-2r}=\frac{A}{r}+\frac{B}{r-2}$ $1=A(r-2)+Br$ $-2A=1\Rightarrow A=-\frac{1}{2}$ $A+B=0\Rightarrow B=\frac{1}{2}$ $\therefore \frac{1}{r^2-2r}=-\frac{1}{2r}+\frac{1}{2(r-2)}$ $\sum_{r=3}^{n}\frac{1}{r^2-2r}$ $=\frac{1}{2}\sum_{r=3}^{n}\left(\frac{1}{r-2}-\frac{1}{r}\right)$ $=\frac{1}{2}\left\{\begin{array}{l} 1 \quad -\frac{1}{3} \\ +\frac{1}{2} \quad -\frac{1}{4} \\ +\frac{1}{3} \quad -\frac{1}{5} \\ +\frac{1}{4} \quad -\frac{1}{6} \\ +\ldots \\ +\frac{1}{n-4}-\frac{1}{n-2} \\ +\frac{1}{n-3}-\frac{1}{n-1} \\ +\frac{1}{n-2}-\frac{1}{n} \end{array}\right\}$ $=\frac{1}{2}\left(\frac{3}{2}-\frac{1}{n-1}-\frac{1}{n}\right)$ (shown)
(ii)	As $n\to\infty$, $\frac{1}{n-1}\to 0$ and $\frac{1}{n}\to 0$. Thus $\sum_{r=3}^{\infty}\frac{1}{r^2-2r}$ is a convergent series. $\sum_{r=3}^{\infty}\frac{1}{r^2-2r}=\frac{3}{4}$

(iii)	$(r-1)^2 = r^2 - 2r + 1 > r^2 - 2r$ $\frac{1}{(r-1)^2} < \frac{1}{r^2-2r}$ for $r > 2$ $\sum_{r=4}^{\infty} \frac{1}{(r-1)^2} < \sum_{r=4}^{\infty} \frac{1}{r^2-2r}$ $= \sum_{r=3}^{\infty} \frac{1}{r^2-2r} - \frac{1}{3^2-2(3)}$ $= \frac{3}{4} - \frac{1}{3}$ $= \frac{5}{12}$ (shown)

2	Surface area of cylindrical piece $= \pi y(2x)$ $= 2\pi xy$ Surface area of 2 hemispherical pieces $= 4\pi\left(\frac{y}{2}\right)^2$ $= \pi y^2$ $C = 2\pi xy(3k) + \pi y^2(5k)$ $= \pi ky(6x+5y)$ Since $V = \pi$, $\pi = \pi\left(\frac{y}{2}\right)^2(2x) + \frac{4}{3}\pi\left(\frac{y}{2}\right)^3$ $6 = 3y^2x + y^3$ $x = \frac{6-y^3}{3y^2}$ $C = \pi ky\left(6\left(\frac{6-y^3}{3y^2}\right)+5y\right)$ $= \pi ky\left(\frac{12}{y^2} - 2y + 5y\right)$ $= \frac{12\pi k}{y} + 3\pi ky^2$

$$\frac{dC}{dy} = -\frac{12\pi k}{y^2} + 6\pi k y = 0$$

$$\frac{12\pi k}{y^2} = 6\pi k y$$

$$y^3 = 2 \qquad \Rightarrow y = \sqrt[3]{2}$$

When $y = \sqrt[3]{2}$, $x = \dfrac{6-2}{3\left(\sqrt[3]{2}\right)^2} = \dfrac{4}{3\left(\sqrt[3]{4}\right)} = \dfrac{\sqrt[3]{16}}{3}$

$$\frac{d^2C}{dy^2} = \frac{24\pi k}{y^3} + 6\pi k = \frac{24\pi k}{2} + 6\pi k = 18\pi k > 0$$

Hence, C is minimum when $y = \sqrt[3]{2}$ and $x = \dfrac{\sqrt[3]{16}}{3}$.

3(a) (i)

$$z = \frac{(1-\mathrm{i})^3}{\sqrt{2}(a+\mathrm{i})^2}$$

$$\Rightarrow |z| = \frac{|1-\mathrm{i}|^3}{\sqrt{2}|a+\mathrm{i}|^2}$$

$$\Rightarrow \frac{1}{2} = \frac{\left(\sqrt{2}\right)^3}{\sqrt{2}|a+\mathrm{i}|^2}$$

$$\Rightarrow |a+\mathrm{i}|^2 = 4$$

$$\Rightarrow |a+\mathrm{i}| = 2$$

$$\Rightarrow \sqrt{a^2+1} = 2$$

$$\Rightarrow a = -\sqrt{3} \text{ (since } a < 0)$$

$$3\arg(1-\mathrm{i}) - \left[\arg\sqrt{2} + 2\arg(-\sqrt{3}+\mathrm{i})\right]$$

$$= 3\left(-\frac{\pi}{4}\right) - 2\left(\frac{5\pi}{6}\right) = -\frac{29\pi}{12}$$

$$\therefore \arg z = -\frac{29\pi}{12} + 2\pi$$

$$= -\frac{5\pi}{12} \text{ (shown)}$$

(a) (ii)

For z^n to have equal real and imaginary parts,

$n\arg z = \dfrac{\pi}{4} + k\pi \Rightarrow -\dfrac{5n\pi}{12} = \pi\left(\dfrac{1}{4} + k\right)$, where $k \in \mathbb{Z}$

$$n = -\frac{3+12k}{5}$$

For smallest integer n, let $k = -4$

$\therefore$ smallest integer $n = 9$

(b)	Method 1: $q=\left(\frac{e^{-i\theta}}{e^{i\theta}-i}\right)\left(\frac{e^{-i\theta}+i}{e^{-i\theta}+i}\right)$ $=\frac{e^{-2i\theta}+ie^{-i\theta}}{2+i\left(e^{i\theta}-e^{-i\theta}\right)}$ $\text{Re}(q)=\frac{\cos 2\theta+\sin\theta}{2-2\sin\theta}$ $=\frac{1-2\sin^2\theta+\sin\theta}{2-2\sin\theta}$ $=\frac{(1+2\sin\theta)(1-\sin\theta)}{2(1-\sin\theta)}$ $=\frac{1}{2}(1+2\sin\theta)$ (shown) Method 2: $q=\frac{e^{-i\theta}}{e^{i\theta}-i}$ $=\frac{\cos\theta-i\sin\theta}{\cos\theta-i(1-\sin\theta)}$ $=\frac{\cos\theta-i\sin\theta}{\cos\theta-i(1-\sin\theta)}\left(\frac{\cos\theta+i(1-\sin\theta)}{\cos\theta+i(1-\sin\theta)}\right)$ $\text{Re}(q)=\frac{\cos^2\theta+\sin\theta(1-\sin\theta)}{\cos^2\theta+(1-\sin\theta)^2}$ $=\frac{\cos^2\theta+\sin\theta-\sin^2\theta}{\cos^2\theta+1-2\sin\theta+\sin^2\theta}$ $=\frac{1-2\sin^2\theta+\sin\theta}{2(1-\sin\theta)}$ $=\frac{(1+2\sin\theta)(1-\sin\theta)}{2(1-\sin\theta)}$ $=\frac{1}{2}(1+2\sin\theta)$ (shown)

4 (i)	If l and p do not intersect, 2 conditions have to be satisfied. **Condition 1:** l is parallel to p Direction vector of l is perpendicular to normal vector of p $\begin{pmatrix}2\\b\\3\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=0$ $2+b-3=0$ $b=1$

	Condition 2: point on l is not on p Point on l does not satisfy plane equation $\begin{pmatrix}-1\\a\\4\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}\neq 11$ $-1+a-4\neq 11$ $a\neq 16$
(ii)	Since B lies on p, $l_{AB}: \underset{\sim}{r}=\begin{pmatrix}-1\\-2\\4\end{pmatrix}+s\begin{pmatrix}1\\1\\-1\end{pmatrix}, s\in\mathbb{R}$ $p: \underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=11$ To find point B, $\left[\begin{pmatrix}-1\\-2\\4\end{pmatrix}+s\begin{pmatrix}1\\1\\-1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\1\\-1\end{pmatrix}=11$ $-1+s-2+s-4+s=11$ $3s=18$ $s=6$ $\overrightarrow{OB}=\begin{pmatrix}-1\\-2\\4\end{pmatrix}+6\begin{pmatrix}1\\1\\-1\end{pmatrix}=\begin{pmatrix}5\\4\\-2\end{pmatrix}$ Coordinates of B are $(5,4,-2)$.
(iii)	Let A' be the reflected point of A in p. $\overrightarrow{OB}=\frac{\overrightarrow{OA'}+\overrightarrow{OA}}{2}$ $\overrightarrow{OA'}=2\begin{pmatrix}5\\4\\-2\end{pmatrix}-\begin{pmatrix}-1\\-2\\4\end{pmatrix}=\begin{pmatrix}11\\10\\-8\end{pmatrix}$ Line of reflection of l in p: $\underset{\sim}{r}=\begin{pmatrix}11\\10\\-8\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\3\end{pmatrix}, \lambda\in\mathbb{R}$
(iv)	Since C is on l, $\overrightarrow{OC}=\begin{pmatrix}-1\\-2\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\3\end{pmatrix}$ for some $\lambda\in\mathbb{R}$ $\overrightarrow{BC}=\begin{pmatrix}-1\\-2\\4\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\3\end{pmatrix}-\begin{pmatrix}5\\4\\-2\end{pmatrix}=\begin{pmatrix}-6\\-6\\6\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\3\end{pmatrix}$

$$\left|\overrightarrow{BC}\right| = \sqrt{164}$$

$$\sqrt{(-6+2\lambda)^2+(-6+\lambda)^2+(6+3\lambda)^2} = \sqrt{164}$$

$$36-24\lambda+4\lambda^2+36-12\lambda+\lambda^2+36+36\lambda+9\lambda^2 = 164$$

$$14\lambda^2-56=0$$

$$\lambda=-2 \quad \text{or} \quad \lambda=2$$

$$\overrightarrow{OC} = \begin{pmatrix}-1\\-2\\4\end{pmatrix} - 2\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}-5\\-4\\-2\end{pmatrix} \quad \text{or} \quad \overrightarrow{OC} = \begin{pmatrix}-1\\-2\\4\end{pmatrix} + 2\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}3\\0\\10\end{pmatrix}$$

(v)

Method I

$$\overrightarrow{AB} = \begin{pmatrix}5\\4\\-2\end{pmatrix} - \begin{pmatrix}-1\\-2\\4\end{pmatrix} = \begin{pmatrix}6\\6\\-6\end{pmatrix}$$

$$\left|\overrightarrow{AB}\right| = \sqrt{3\left(6^2\right)} = \sqrt{108}$$

$$\sin \measuredangle BCA = \frac{\sqrt{108}}{\sqrt{164}}$$

$\measuredangle BCA = 54.2°$ (1 d.p.)

Method 2

B and the 2 points of C form an isosceles triangle.

$\therefore \measuredangle BCA =$ acute angle between $\overrightarrow{BC}$ and l

$$\lambda = 2 \Rightarrow \overrightarrow{BC} = \begin{pmatrix}-6\\-6\\6\end{pmatrix} + 2\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}-2\\-4\\12\end{pmatrix}$$

$$\cos \measuredangle BCA = \frac{\left|\begin{pmatrix}-2\\-4\\12\end{pmatrix} \cdot \begin{pmatrix}2\\1\\3\end{pmatrix}\right|}{\sqrt{164}\sqrt{2^2+1^2+3^2}}$$

$$= \frac{28}{\sqrt{164}\sqrt{14}}$$

$\measuredangle BCA = 54.2°$ (1 d.p.)

Method 3

Consider vectors $\overrightarrow{BC}$ & $\overrightarrow{AC}$.

$$\lambda = 2 \Rightarrow \overrightarrow{BC} = \begin{pmatrix}-6\\-6\\6\end{pmatrix} + 2\begin{pmatrix}2\\1\\3\end{pmatrix} = \begin{pmatrix}-2\\-4\\12\end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix}3\\0\\10\end{pmatrix} - \begin{pmatrix}-1\\-2\\4\end{pmatrix} = \begin{pmatrix}4\\2\\6\end{pmatrix}$$

$$\cos \measuredangle BCA = \frac{\begin{pmatrix}-2\\-4\\12\end{pmatrix} \cdot \begin{pmatrix}4\\2\\6\end{pmatrix}}{\sqrt{164}\sqrt{4^2+2^2+6^2}}$$

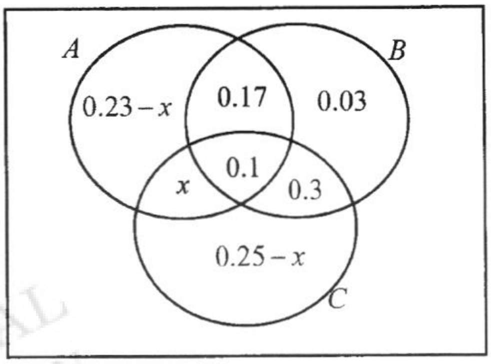
$$= \frac{56}{\sqrt{164}\sqrt{56}}$$

	$\measuredangle BCA = 54.2°$ (1 d.p.)

5 **(i)**	Method 1 (Direct Method) Case 1: Students from Class A are grouped as 2,2,0 Number of ways = $\dfrac{{}^4C_2\,{}^8C_2 \times {}^6C_2}{2!} = 1260$ Case 2: Students from Class A are grouped as 2,1,1 Number of ways = $\dfrac{{}^4C_2\,{}^8C_2 \times {}^2C_1\,{}^6C_3}{2!} = 3360$ Total number of ways = 4620 -- Method 2 (Using complement) Case 1: Students from Class A are grouped as 3,1,0 Number of ways = ${}^4C_3\,{}^8C_1 \times {}^7C_3 = 1120$ Case 2: Students from Class A are grouped in 4,0,0 Number of ways = $\dfrac{{}^4C_4 \times {}^8C_4}{2!} = 35$ Number of ways = $\dfrac{{}^{12}C_4\,{}^8C_4\,{}^4C_4}{3!} - (1120+35) = 4620$
(ii)	No of ways = $(6-1)! \times {}^6C_4 \times 4!$ $= 43200$

6 **(i)**	$\mathrm{P}(A\|B) = \dfrac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$ $0.45 = \dfrac{\mathrm{P}(A \cap B)}{0.6}$ $\mathrm{P}(A \cap B) = 0.27$ $\mathrm{P}(A \cap B \cap C') = 0.27 - 0.1 = 0.17$
(ii)	Since A and C are independent, A and C' are independent, $\mathrm{P}(A \cap C') = \mathrm{P}(A) \times \mathrm{P}(C') = 0.5 \times (1-0.65) = 0.175$ $\mathrm{P}(A \cup C') = \mathrm{P}(A) + \mathrm{P}(C') - \mathrm{P}(A \cap C')$ $= 0.5 + (1-0.65) - 0.175$ $= 0.675$

	Let $\mathrm{P}(A \cap B' \cap C)$ be x.



$\mathrm{P}(A' \cap B' \cap C')$
$= 1-(0.23-x)-0.17-0.1-0.3-0.03-x-(0.25-x)$
$= x-0.08$

Consider:

$x-0.08 \geq 0$ and $0.23-x \geq 0$ and $x \geq 0$ and $0.25-x \geq 0$

$x \geq 0.08$ and $x \leq 0.23$ and $x \geq 0$ and $x \leq 0.25$

Therefore, $0.08 \leq x \leq 0.23$.

When $x = 0.08$, $\mathrm{P}(A' \cap B' \cap C') = 0.08-0.08 = 0$ (least)

When $x = 0.23$, $\mathrm{P}(A' \cap B' \cap C') = 0.23-0.08 = 0.15$ (greatest)

Alternative

Least $\mathrm{P}(A' \cap B' \cap C')$ occurs when x is minimised:

Suppose $x = 0$, then $\mathrm{P}(A \cup B \cup C) = 0.6+0.23+0.25 = 1.08$ which is impossible as $\mathrm{P}(A \cup B \cup C) \leq 1$.

Greatest $\mathrm{P}(A \cup B \cup C) = 1$ and least $\mathrm{P}(A' \cap B' \cap C') = 0$

Greatest $\mathrm{P}(A' \cap B' \cap C')$ occurs when x is maximised:

$0.23-x = 0$
$x = 0.23$
$\mathrm{P}(A' \cap B' \cap C') = 1-0.23-0.02-0.6 = 0.15$
Greatest $\mathrm{P}(A' \cap B' \cap C') = 0.15$

7 (i)	$\mathrm{P}(X=9) = \mathrm{P}(3,3,3) = \frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} = \frac{1}{21}$ $\mathrm{P}(X=8) = \mathrm{P}(3,3,2) = \frac{4}{9} \times \frac{3}{8} \times \frac{3}{7} \times 3 = \frac{3}{14}$ $\mathrm{P}(X=7) = \mathrm{P}(3,3,1)+\mathrm{P}(3,2,2) = \left(\frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} \times 3\right) + \left(\frac{4}{9} \times \frac{3}{8} \times \frac{2}{7} \times 3\right) = \frac{2}{7}$

	$P(X=6)=P(3,2,1)+P(2,2,2)=\left(\frac{4}{9}\times\frac{3}{8}\times\frac{2}{7}\times 3!\right)+\left(\frac{3}{9}\times\frac{2}{8}\times\frac{1}{7}\right)=\frac{25}{84}$ $P(X=5)=P(3,1,1)+P(2,2,1)=\left(\frac{4}{9}\times\frac{2}{8}\times\frac{1}{7}\times 3\right)+\left(\frac{3}{9}\times\frac{2}{8}\times\frac{2}{7}\times 3\right)=\frac{5}{42}$ $P(X=4)=P(2,1,1)=\frac{3}{9}\times\frac{2}{8}\times\frac{1}{7}\times 3=\frac{1}{28}$ **OR** $P(X=9)=\frac{{}^4C_3}{{}^9C_3}=\frac{1}{21}$ $P(X=8)=\frac{{}^4C_2\times{}^3C_1}{{}^9C_3}=\frac{3}{14}$ $P(X=7)=\frac{{}^4C_2\times{}^2C_1}{{}^9C_3}+\frac{{}^4C_1\times{}^3C_2}{{}^9C_3}=\frac{2}{7}$ $P(X=6)=\frac{{}^4C_1\times{}^3C_1\times{}^2C_1}{{}^9C_3}+\frac{{}^3C_3}{{}^9C_3}=\frac{25}{84}$ $P(X=5)=\frac{{}^4C_1\times{}^2C_2}{{}^9C_3}+\frac{{}^3C_2\times{}^2C_1}{{}^9C_3}=\frac{5}{42}$ $P(X=4)=\frac{{}^2C_2\times{}^3C_1}{{}^9C_3}=\frac{1}{28}$

x	4	5	6	7	8	9
$P(X=x)$	$\frac{1}{28}$	$\frac{5}{42}$	$\frac{25}{84}$	$\frac{2}{7}$	$\frac{3}{14}$	$\frac{1}{21}$

(ii)	$P(X \text{ is odd})=\frac{1}{21}+\frac{2}{7}+\frac{5}{42}=\frac{19}{42}$ $E(Y)=\left(1-\frac{19}{42}\right)(-w)+\frac{19}{42}(w)$ $-0.8=-\frac{23}{42}w+\frac{19}{42}w$ $w=8.4$ $\text{Var}(Y)=E(Y^2)-[E(Y)]^2$ $=\left[\left(1-\frac{19}{42}\right)(-8.4)^2+\left(\frac{19}{42}\right)(8.4)^2\right]-(-0.8)^2$ $=69.92$
(iii)	Player wins \$2w $\Rightarrow$ wins 6 out of 10 rounds Let R be the number of rounds that the player wins, out of 10 rounds. $R \sim B\left(10,\frac{19}{42}\right)$

	$\text{P}(R=6)=0.162$ (3 s.f.) **Alternative** Required probability $=\left(\frac{19}{42}\right)^6\left(1-\frac{19}{42}\right)^4\times\frac{10!}{6!4!}$ $=0.162$ (3 s.f.)

8 (i)	Let X denote the waiting time for vaccination in Polyclinics. $X\sim\text{N}\left(25,6^2\right)$ $\text{P}(X>30)=0.20233$ $\text{P}(X<30)=1-0.20233=0.79767$ Required probability $=\text{P}(X>30)\times\text{P}(X<30)\times 2!$ $=0.323$ (3 s.f.)
(ii)	Let Y denote the waiting time for vaccination in Community Clubs. $Y\sim\text{N}\left(20,3^2\right)$ $X-Y\sim\text{N}\left(25-20,6^2+3^2\right)$ $\therefore X-Y\sim\text{N}(5,45)$ $\text{P}(\lvert X-Y\rvert\geq 3)$ $=\text{P}(X-Y\geq 3)+\text{P}(X-Y\leq -3)$ $=0.61720+0.11652$ $=0.734$ (3 s.f.)
(iii)	Let $A=\frac{X_1+\cdots+X_5+Y_1+\cdots+Y_{15}}{20}$ $\text{E}(A)=\text{E}\left(\frac{X_1+\cdots+X_5+Y_1+\cdots+Y_{15}}{20}\right)$ $=\frac{1}{20}\text{E}\left(X_1+\cdots+X_5+Y_1+\cdots+Y_{15}\right)$ $=\frac{1}{20}(5(25)+15(20))$ $=21.25$

	$\text{Var}(A)=\text{Var}\left(\frac{X_1+\cdots+X_5+Y_1+\cdots+Y_{15}}{20}\right)$ $=\left(\frac{1}{20}\right)^2\text{Var}\left(X_1+\cdots+X_5+Y_1+\cdots+Y_{15}\right)$ $=\left(\frac{1}{20}\right)^2\left[\text{Var}(X_1)+...+\text{Var}(X_5)+\text{Var}(Y_1)+...+\text{Var}(Y_{15})\right]$ $=\frac{1}{20^2}\left(5\left(6^2\right)+15\left(3^2\right)\right)$ $=0.7875$ $\therefore A\sim\text{N}(21.25, 0.7875)$ $\text{P}(A<21)=0.389$ (3 s.f.) Required assumption: The waiting times for vaccinations of the people in Polyclinics and Community Clubs are all mutually independent with one another.

9 **(i)**	An unbiased estimate of μ is: $\bar{x}=\frac{36}{50}+300=300.72$ An unbiased estimate of σ^2 is: $s^2=\frac{1}{49}\left(612-\frac{36^2}{50}\right)$ $=11.961$ $=12.0$ (3 s.f.)
(ii)	H_0 : $\mu=300$ H_1 : $\mu>300$ Under H_0, since $n=50$ is large, by Central Limit Theorem, $\overline{X}\sim\text{N}\left(300,\frac{11.961}{50}\right)$ approximately. The test statistic is $Z=\frac{\overline{X}-300}{\sqrt{11.961}/\sqrt{50}}\sim\text{N}(0,1)$ approximately. From GC, p-value = 0.0705 (3 s.f.) Since the p-value = 0.0705 > 0.05, we **do not reject H_0** and conclude that there is **insufficient** evidence at the 5% level that the mean number of calories is more than 300 (*or* the nutritionist's belief is correct *or* the mean number of calories has been understated).
(iii)	Since sample size is large, by Central Limit Theorem, the sample mean number of calories for a packet of chocolate will be approximately normal.
(iv)	There is a probability of 0.05 that the test will show that the mean number of calories for a packet of chocolate is more than 300, when it is in fact 300.

(v)	$s^2 = \frac{40}{39}(10) = 10.256$ H_0 : $\mu = 300$ H_1 : $\mu \neq 300$ Under H_0, since $n = 40$ is large, by Central Limit Theorem, $\overline{X} \sim N\left(300, \frac{10.256}{40}\right)$ approximately. The test statistic is $Z = \frac{\overline{X} - 300}{\sqrt{10.256}/\sqrt{40}} \sim N(0, 1)$ approximately. Critical Region: $z \leq -2.1701$ or $z \geq 2.1701$ Given that H_0 is rejected, $\frac{k-300}{\sqrt{10.256}/\sqrt{40}} \leq -2.1701$ or $\frac{k-300}{\sqrt{10.256}/\sqrt{40}} \geq 2.1701$ $k - 300 \leq -2.1701\left(\frac{\sqrt{10.256}}{\sqrt{40}}\right)$ or $k - 300 \geq 2.1701\left(\frac{\sqrt{10.256}}{\sqrt{40}}\right)$ $k \leq 299$ (3 s.f.) or $k \geq 301$ (3 s.f.)

10 (i)	Let X be the number of defective bottles, out of 30 bottles. Then, $X \sim B(30, 0.04)$ P(accepting a batch of bottles) $= P(X_1 = 0) + P(X_1 = 1)P(X_2 = 0) + P(X_1 = 2)P(X_2 \leq 1)$ $= 0.54853$ $= 0.549$ (3 s.f.)
(ii)	Required probability = P(at most 2 defective bottles | batch is accepted) $= \frac{0.54853 - P(X_1 = 2)P(X_2 = 1)}{0.54853}$ $= \frac{0.46701}{0.54853}$ $= 0.851$ (3 s.f.)
(iii)	Exactly 20 fewer defective bottles than non-defective bottles means $X = 5$ $P(X = 5) = 0.0052591$ $= 0.00526$ (3 s.f.)
(iv)	Let Y be the number of boxes, out of n, with exactly 20 fewer defective bottles than non-defective bottles. $Y \sim B(n, P(X = 5))$ ie. $Y \sim B(n, 0.0052591)$

	$P(Y \geq 2) \leq 0.05$ $1 - P(Y \leq 1) \leq 0.05$ From GC, <table><tr><td>n</td><td>$1 - P(Y \leq 1)$</td></tr><tr><td>67</td><td>0.0488</td></tr><tr><td>68</td><td>0.0501</td></tr></table> Greatest possible value of $n = 67$
(v)	$X \sim B(30, 0.04)$ $E(X) = 30(0.04) = 1.2$ $Var(X) = 30(0.04)(0.96) = 1.152$ Since sample size = 50 is large, by Central Limit Theorem, $\bar{X} \sim N\left(1.2, \frac{1.152}{50}\right)$ approximately $\bar{X} \sim N(1.2, 0.02304)$ Required probability = $P(\bar{X} < 1) = 0.0938$ (3 s.f.)