1 **Do not use a calculator in answering this question.**

It is given that $2-\mathrm{i}$ is a root of the equation $2z^3+az^2-2z+b=0$.

(a) Find the values of the real numbers a and b and the remaining roots of the equation. [4]

(b) Using these values of a and b, deduce the roots of the equation

$$bz^3-2z^2+az+2=0.$$ [2]

2 Figure 1 shows a sketch of the curve $y=\mathrm{f}(x)$ and A is the region under the curve between $x=1$ and $x=3$. Yvonne and Irene use different ways to draw 4 rectangles of equal width, h, to estimate the area of A.

Figure 2 shows 4 rectangles drawn by Yvonne, with the curve intersecting each rectangle at the mid-point of its width. The x-coordinates of the mid-points are m_1, m_2, m_3 and m_4.

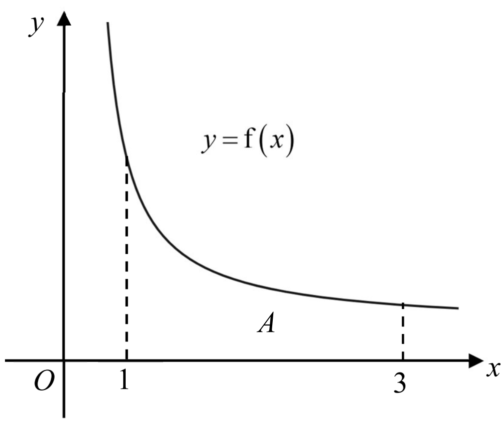


Figure 1

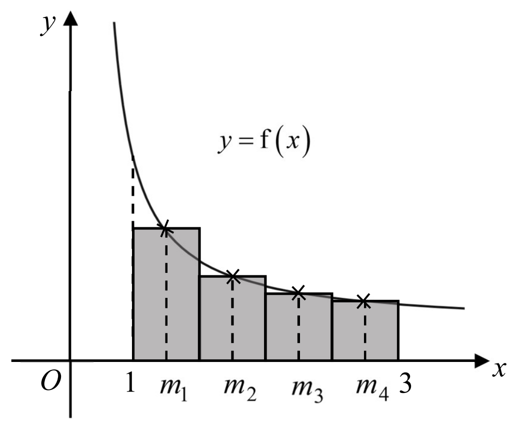


Figure 2

(a) State the values of h, m_1, m_2, m_3 and m_4. [2]

(b) The sum of area of rectangles using Yvonne's method is denoted by B. Show that $B=h\sum_{r=0}^{3}\left(\mathrm{f}\left(a+rh\right)\right)$, where the value of a is to be determined. [2]

(c) Irene finds that the sum of area of 4 rectangles that she has drawn is $C=h\sum_{r=0}^{3}\left(\mathrm{f}\left(1+rh\right)\right)$. Draw these rectangles in **Figure 1**. [1]

You are now given that $\mathrm{f}(x)=\dfrac{1}{x}+1$.

(d) By finding the numerical values of B, C and the actual area of region A, explain how these values **and** the rectangles drawn in Figures 1 and 2 show that Yvonne's estimation of the area of A is better than Irene's. [2]

3 **(a)** On the same axes, sketch the graphs of $y=\left|x(x-5)\right|$ and $y=\sqrt{2}\left|x-5\right|$. [2]

(b) Hence, or otherwise, solve exactly the inequality $\left|x(x-5)\right|>\sqrt{2}\left|x-5\right|$. [4]

4 The points P, Q and R have position vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ respectively where $\mathbf{p}$ and $\mathbf{q}$ are non-zero and non-parallel. The points P and Q are fixed and R varies.

(a) Given that $\mathbf{r}\times\mathbf{q}=-\mathbf{q}\times\mathbf{p}$, describe geometrically the set of all possible positions of the point R. [4]

(b) Given instead that $\mathbf{r}=\begin{pmatrix}x\\y\\z\end{pmatrix}, \mathbf{p}=\begin{pmatrix}1\\-2\\3\end{pmatrix}, \mathbf{q}=\begin{pmatrix}q_1\\q_2\\q_3\end{pmatrix}$, and that $\mathbf{q}\cdot(\mathbf{p}-\mathbf{r})=0$, find the relationship between x, y, and z in terms of q_1, q_2 and q_3. Describe the set of all possible positions of the point R in this case. [3]

(c) It is now given that $\left|\mathbf{q}\right|=1$ and C is a point with position vector $\mathbf{c}$ such that $\mathbf{q}\cdot(\mathbf{p}-\mathbf{c})\neq 0$. Give a geometrical meaning of $\left|\mathbf{q}\cdot(\mathbf{p}-\mathbf{c})\right|$. [1]

5 **(a)** Show that $\dfrac{1}{(r-1)!}-\dfrac{2}{r!}+\dfrac{1}{(r+1)!}=\dfrac{\mathrm{f}(r)}{(r+1)!}$, where $\mathrm{f}(r)$ is a function in r to be found. [1]

The sum $\displaystyle\sum_{r=2}^{N}\frac{\mathrm{f}(r)}{(r+1)!}$ is denoted by S_N.

(b) Using your answer in part **(a)**, find S_N in terms of N. [3]

(c) Give a reason why S_N converges and find the exact value of S_∞. [2]

(d) Find the smallest value of N such that S_N is within 10^{-7} of S_∞. [2]

6 **(a)** Show that $1+\mathrm{e}^{-\mathrm{i}\alpha}=2\cos\dfrac{\alpha}{2}\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}$, where $-\pi<\alpha\leq\pi$. [2]

(b) Hence or otherwise, show that

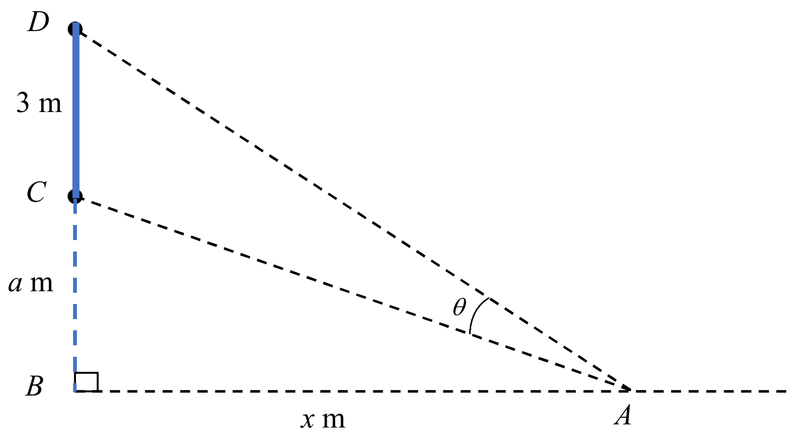
$$\left(1+\mathrm{e}^{-\mathrm{i}\alpha}\right)^3-\left(1+\mathrm{e}^{\mathrm{i}\alpha}\right)^3=-16\mathrm{i}\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{3\alpha}{2}\right).$$ [3]

(c) Given further that $0<\alpha<\dfrac{2}{3}\pi$ and $z=\left(1+\mathrm{e}^{-\mathrm{i}\alpha}\right)^3-\left(1+\mathrm{e}^{\mathrm{i}\alpha}\right)^3$, deduce the modulus and argument of z. Express your answers in terms of α whenever applicable. [2]

7 **(a)** It is given that $\mathrm{f}(x)=\ln(1+\sin 2x)+2$, where $0 \le x \le \dfrac{\pi}{2}$. By using differentiation, find $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$. Hence write down the Maclaurin series for $\mathrm{f}(x)$, up to and including the term in x^2. [5]

(b) Given that x is a sufficiently small angle, find the series expansion of $\dfrac{1}{\cos 2x+\sin x}$, up to and including the term in x^2. [4]

8



The diagram shows a cross-sectional view of an advertisement sign *CD* hung against a vertical wall, where the point *C* is *a* metres above the eye level *AB* of an observer who is *x* metres from *B*. The distance *CD* is 3 metres and angle *CAD* is θ.

(a) By expressing θ as the difference of two angles, or otherwise, show that

$$\tan\theta=\frac{3x}{x^2+3a+a^2}\ .$$ [3]

(b) Find, in terms of *a*, the value of *x* which maximises $\tan\theta$, simplifying your answer. Find also the corresponding value of $\tan\theta$. You do not need to show that $\tan\theta$ is maximum. [4]

(c) Find $\tan ADB$ when $\tan\theta$ is maximum, expressing your answer in terms of *a*. Find the approximate value of angle *ADB* when *a* is much greater than 3. [3]

9 A curve C has parametric equations

$$x = a\cos 2t, \qquad y = 2a\cos t,$$

for $0 \le t \le \frac{\pi}{2}$, where a is a positive constant.

(a) Show that the equation of normal to the curve at the point $P\left(a\cos 2p,\ 2a\cos p\right)$ is

$$y = -2\cos p\left(x - 2a\cos^2 p\right). \qquad [3]$$

(b) The normal at P meets the x-axis at the point R. Show that the area enclosed by the x-axis, the normal at P and C is given by

$$4a^2 \int_{t_1}^{t_2} \cos t \sin 2t \ \mathrm{d}t + a^2 \cos p,$$

where the values of t_1 and t_2 should be stated. [6]

.

(c) Hence find in terms of a, the exact area in part **(b)** given now that $p = \frac{\pi}{3}$. [3]

10 Alan and Betty bought an apartment at \$450, 000. They are eligible to take a housing loan, up to 85% of the cost of the apartment, for a maximum of 30 years.

After careful consideration, the couple decides to borrow 85% of the cost of the apartment. They will make a cash repayment of \$$x$ at the beginning of each month, starting 1st July 2022. Interest will be charged with effect from 31^{st} July 2022 at a monthly interest rate of 0.2% for the remaining amount owed at the end of each month.

(a) Find the amount of money owed on 31^{st} of July 2022 after the interest for the month has been added. Express your answer in terms of x. [1]

(b) Show that the total amount of money owed after the nth repayment at the beginning of the month is

$$1.002^{n-1}(382500) - 500x(1.002^n - 1). \qquad [4]$$

(c) Find the earliest date on which the couple will be able to pay off the loan completely if $x = 2000$, and state the amount of repayment on this date. [4]

(d) If the couple wishes to pay off the loan completely on 1^{st} Jan 2050 (after the repayment on this day), what should the monthly repayment be? [3]

11 Naturalists are managing a wildlife reserve to increase the number of plants of a rare species. The number of plants at time t years is denoted by N, where N is treated as a continuous variable. It is given that the rate of increase of N with respect to t is proportional to $(N-120)$.

(a) Write down a differential equation relating N and t. [1]

Initially, the number of plants was 600. It is noted that at a time when there were 750 plants, the number of plants was increasing at a rate of 63 per year.

(b) Express N in terms of t. [6]

(c) The naturalist has a target of increasing the number of plants from 600 to 2500 within 15 years. Justify whether this target will be met. [2]

Alongside the monitoring of the number of plants of this rare species, naturalists also study the rate of increase of its height, h cm, with respect to time, t years after planting. The height of a plant is modelled by the differential equation

$$\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{1}{2}\sqrt{\left(24-\frac{1}{3}h\right)}.$$

The plant is planted as a seedling of negligible height, so that $h=0$ when $t=0$.

(d) State the maximum height of the plant, according to this model. [1]

(e) Find an expression for t in terms of h, and hence find the time the plant takes to reach 24 cm. [5]

Section A: Pure Mathematics [40 marks]

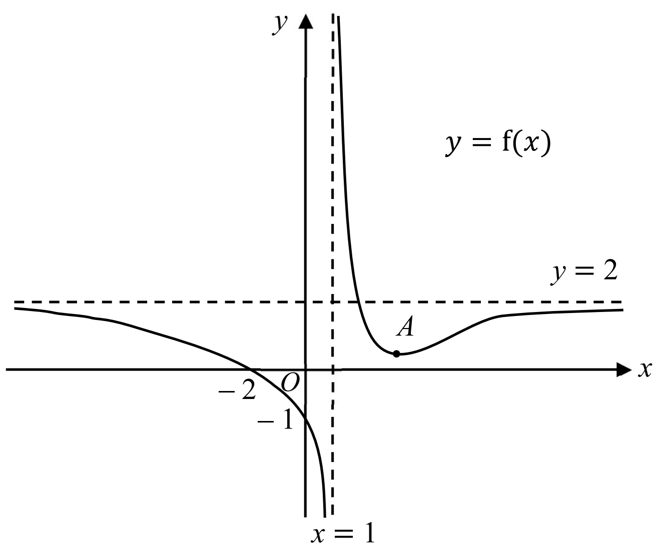
1 It is given that $I = \int \frac{x}{\sqrt{4-2x}}\, dx$.

(a) Use integration by parts to find an expression for I. [2]

(b) Use the substitution $u = 4-2x$ to find another expression for I. [2]

(c) Show algebraically that the answers to parts **(a)** and **(b)** differ by a constant. [2]

2



The diagram shows the curve $y = f(x)$ with a turning point $A\left(3, \frac{1}{2}\right)$. The curve crosses the axes at $x = -2$ and $y = -1$ and the lines $x = 1$ and $y = 2$ are the asymptotes of the curve.

Sketch the following curves on separate diagrams, stating, if it is possible to do so, the equations of any asymptotes and the coordinates of any points where each curve crosses the axes and of any turning points.

(a) $y = f'(x)$, [2]

(b) $y = \frac{1}{f(x)}$. [3]

3 Functions f and g are defined by

$$\mathrm{f}: x \mapsto x^2 + 3x - 1, \ x \in \mathbb{R}, \ x \leq k,$$

$$\mathrm{g}: x \mapsto \sqrt{x+5}, \ x \in \mathbb{R}, \ x \geq -5.$$

(a) Given that f^{-1} exists, state the largest possible value of k. Using this value of k, find $\mathrm{f}^{-1}(x)$. [3]

For the rest of this question, let $k = -2$.

(b) Find the exact solution of the equation $\mathrm{f}(x) = \mathrm{f}^{-1}(x)$. [2]

(c) Determine whether the composite functions fg and gf exist. If the composite function exists, give a definition (including the domain) of the function. [3]

(d) Hence find the exact range of the composite function that exists. [1]

4 **(a)** State a sequence of transformations that will transform the curve with equation $y = \ln x$ onto the curve with equation $y = \ln\,(2x+3)^3$. [3]

A curve has equation $y = \mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} \ln 64 & \text{for } x > \frac{1}{2}, \\ \ln\,(2x+3)^3 & \text{for } -\frac{1}{2} \leq x \leq \frac{1}{2}, \\ \ln 8 & \text{otherwise.} \end{cases}$$

(b) Sketch the curve for $-1 \leq x \leq 1$. [3]

(c) Find the numerical value of the volume generated when the region bounded by the curve $y = \mathrm{f}(x)$, the line $x = 1$ and the line $y = \ln 27$ is rotated completely about the y-axis. Give your answer correct to 3 decimal places. [3]

5 The plane p has equation $\mathbf{r}=\begin{pmatrix}6\\2\\-5\end{pmatrix}+\lambda\begin{pmatrix}0\\3\\-1\end{pmatrix}+\mu\begin{pmatrix}2\\-2\\1\end{pmatrix}$, where λ and μ are parameters.

The line l passes through the points A and B with position vectors $4\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and $4\mathbf{j}+2\mathbf{k}$ respectively.

(a) Find the coordinates of the point of intersection between p and l. [5]

(b) Find the cartesian equations of the planes such that the perpendicular distance from each plane to p is $\sqrt{41}$. [3]

Another line m has equation $\frac{1-x}{3}=y+2=\frac{z-3}{a}$.

(c) Find the value of a such that p and m do not meet in a unique point. [3]

Section B: Probability and Statistics [60 marks]

6 A group of 12 people consists of 5 men and 7 women. One of the women is the wife of one of the men.

(a) How many committees of 5 can be formed which include at least 3 women? [2]

(b) The 12 people sit at random at a round table. Find the probability that the husband and his wife are seated together and no two men are next to each other. [3]

7 Bag A contains 4 balls numbered 3, 5, 6 and 9. Bag B contains 5 balls numbered 1, 2, 7, 9 and 9. Bag C contains 8 balls numbered 3, 4, 4, 8, 8, 9, 9 and 9. All the balls are indistinguishable apart from the number on the balls. One ball is selected at random from each bag.

- X is the event that exactly two of the selected balls have the same number.
- Y is the event that the ball selected from bag A is numbered 5.

(a) Show that $\mathrm{P}(X)=\frac{21}{80}$. [2]

(b) Find $\mathrm{P}(X\cap Y)$ and hence determine whether X and Y are independent. [3]

(c) Find the probability that one ball is numbered 7, given that exactly two of the selected balls have the same number. [2]

8 A fair cubical die has faces each labelled with one of the four distinct numbers: a, $2a$, b and $3b$. The die is thrown once and the number on the uppermost face is the score T.

It is given that the mode of T is a and that $\mathrm{P}(T=2a)=\mathrm{P}(T=3b)=\frac{1}{6}$.

(a) Draw a table to show the probability distribution of T. [2]

(b) Given that the mean score is $\frac{25}{6}$, find the variance of the score in terms of a. [5]

9 A flower shop makes 75 bouquets of flowers daily. On average, p% of the bouquets have LED lights. Assume that X, the number of bouquets of flowers with LED lights made daily, follows a binomial distribution.

(a) Given that there is a probability of 0.0288 that fewer than 2 bouquets made in a day have LED lights, write down an equation in terms of p and hence find p correct to 4 decimal places. [3]

It is now given that $p = 7.5$.

(b) Find the most likely number of bouquets with LED lights made in a day. [2]

(c) 30 days are randomly selected. Find the probability that the mean number of bouquets with LED lights made per day is at least 5. [3]

10 In an experiment, a chemist applied different quantities, x ml, of a chemical to 7 samples of a type of metal, and the times, t hours, for the metal to discolour were measured. The results are given in the table.

x	1.2	2.0	2.7	3.8	4.8	5.6	7.0
t	2.2	4.5	5.8	7.3	8.0	9.0	10.5

(a) Draw a scatter diagram for these values, labelling the axes. [1]

(b) Find, correct to 4 decimal places, the product moment correlation coefficient between

(i) $\ln x$ and t,

(ii) e^{-x} and t. [2]

(c) Explain which of the two cases in part **(b)** is more appropriate and find the equation of a suitable regression line for this case. [3]

(d) Use the equation of your regression line to estimate the value of the quantity of chemical applied to the metal when the time taken for the metal to discolour is 8.5 hours. Explain whether your estimate is reliable. [2]

11 Farm A claims that the duck eggs from their farm have a mean mass of 70 grams. A random sample of 50 duck eggs is selected. The masses, x grams, are summarised as follows.

$$\sum(x-70)=186.35, \quad \sum(x-70)^2=10494.$$

(a) Calculate unbiased estimates of the population mean and variance. [2]

(b) Test, at the 5% level of significance, whether Farm A's claim is valid. [4]

(c) State, with a reason, whether it is necessary to assume a normal distribution for the test to be valid. [1]

(d) Explain the meaning of 'at the 5% level of significance' in the context of the question. [1]

Farm B claims that their duck eggs have a mean mass of more than 70 grams. A random sample of 40 duck eggs is taken, and it is found that their mean mass and variance are k grams and 146 grams^2 respectively. Given that a test at the 3% significance level indicates that Farm B's claim is valid, find the set of values of k. [4]

12 **In this question you should state the parameters of any distributions that you use.**

A supermarket sells two types of sugar. White sugar is sold in packets with the labelled mass of 1 kg. The mass of a packet of white sugar may be regarded as a normally distributed random variable with mean 1.05 kg and standard deviation 0.03 kg.

(a) What mass is exceeded by 80% of the packets of white sugar? Give your answer correct to 3 decimal places. [1]

(b) A packet of white sugar that weighs less than 98% of the labelled mass is considered underweight. Find the probability that at most 1 out of 10 randomly chosen packets of white sugar is underweight. [3]

The masses of packets of brown sugar are normally distributed with mean m kg and standard deviation 0.05 kg and the masses of packets of white sugar and brown sugar have independent normal distributions.

(c) Given that the probability that the total mass of 4 randomly chosen packets of white sugar exceeds twice the mass of a randomly chosen packet of brown sugar is 0.15, find m. [5]

It is now given that $m=2.03$.

(d) Find the probability that the average mass of 4 randomly chosen packets of white sugar and 5 randomly chosen packets of brown sugar exceeds 1.6 kg. [4]

Paper 1 Solution with Markers' Comments

<table>
<tr><td>1</td><td>
Since the coefficients of the cubic equation are all real and $2-\mathrm{i}$ is a root of the equation, another root would be $2+\mathrm{i}$.

<u>Method 1</u>
$$\begin{aligned}(z-2+\mathrm{i})(z-2-\mathrm{i}) &= (z-2)^2-(\mathrm{i})^2\\ &= z^2-4z+4+1\\ &= z^2-4z+5\end{aligned}$$
$$2z^3+az^2-2z+b=0$$
$$\left(z^2-4z+5\right)\left(2z+\frac{b}{5}\right)=0 \quad \text{(By inspection)}$$
Comparing coefficients of z: $-2=-\frac{4b}{5}+10 \Rightarrow b=15$
Comparing coefficients of z^2: $a=\frac{b}{5}-8=-5$
The other roots are $z=2+\mathrm{i}$ and $z=-\frac{3}{2}$.
</td></tr>
<tr><td></td><td>
<u>Method 2 (Not recommended)</u>
Since $2-\mathrm{i}$ is a root to the equation,
$$\begin{aligned}2(2-\mathrm{i})^3+a(2-\mathrm{i})^2-2(2-\mathrm{i})+b&=0\\ 2(8-12\mathrm{i}-6+\mathrm{i})+a(4-4\mathrm{i}-1)-2(2-\mathrm{i})+b&=0\\ 4-22\mathrm{i}+3a-4a\mathrm{i}-4+2\mathrm{i}+b&=0\\ (3a+b)-(20+4a)\mathrm{i}&=0\end{aligned}$$
Comparing the imaginary part: $20+4a=0 \Rightarrow a=-5$
Comparing the real part: $3a+b=0 \Rightarrow b=15$

Since the coefficients of the cubic equation are all real and $2-\mathrm{i}$ is a root of the equation, another root would be $2+\mathrm{i}$.
$$\begin{aligned}(z-2+\mathrm{i})(z-2-\mathrm{i}) &= (z-2)^2-(\mathrm{i})^2\\ &= z^2-4z+4+1\\ &= z^2-4z+5\end{aligned}$$
$$2z^3-5z^2-2z+15=0$$
$$\left(z^2-4z+5\right)(2z+3)=0$$
The other roots are $z=2+\mathrm{i}$ and $z=-\frac{3}{2}$.
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<tr><td></td><td>
$$bz^3-2z^2+az+2=0.$$
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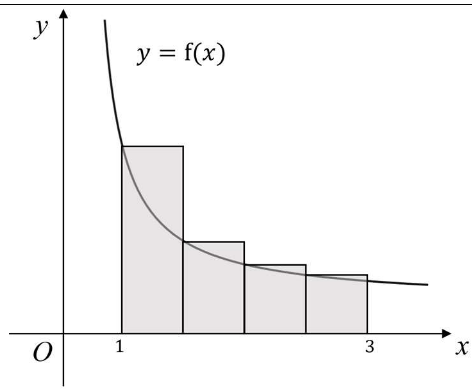
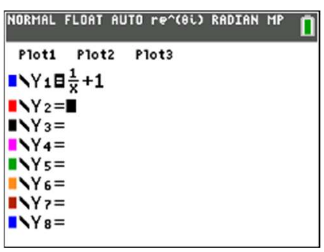
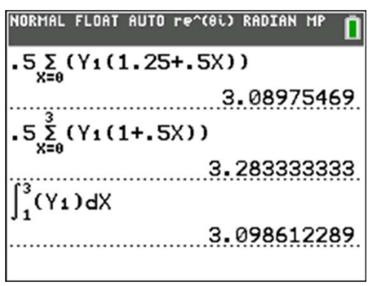
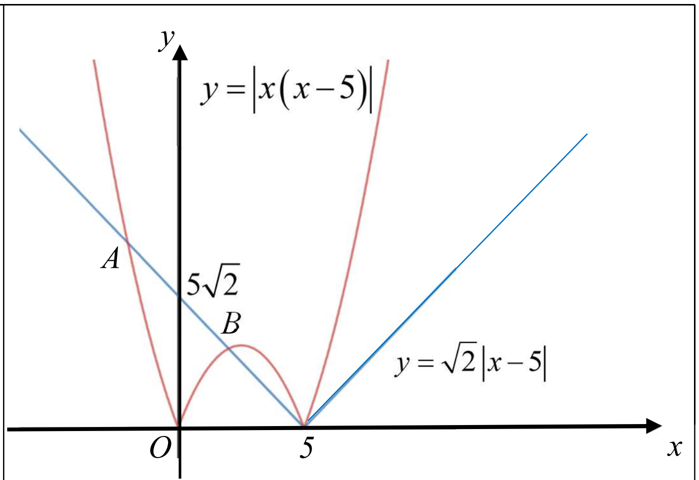
</table>

	Divide z^3 throughout, $b-2\left(\frac{1}{z}\right)+a\left(\frac{1}{z^2}\right)+2\left(\frac{1}{z^3}\right)=0$ $2\left(\frac{1}{z^3}\right)+a\left(\frac{1}{z^2}\right)-2\left(\frac{1}{z}\right)+b=0$ Replace z with $\frac{1}{z}$, $\frac{1}{z}=2-\mathrm{i}$, $\frac{1}{z}=2+\mathrm{i}$ or $\frac{1}{z}=-\frac{3}{2}$ $z=\frac{1}{2-\mathrm{i}}\times\frac{2+\mathrm{i}}{2+\mathrm{i}}$ $z=\frac{1}{2+\mathrm{i}}\times\frac{2-\mathrm{i}}{2-\mathrm{i}}$ $z=-\frac{2}{3}$ $=\frac{2+\mathrm{i}}{5}$ $=\frac{2-\mathrm{i}}{5}$ $\therefore z=\frac{2}{5}+\frac{1}{5}\mathrm{i},\ \frac{2}{5}-\frac{1}{5}\mathrm{i}$ or $-\frac{2}{3}$

2a	$h=\dfrac{3-1}{4}=0.5$ Since m_i are the mid-points of the intervals, $m_1=1.25,\ m_2=1.75,\ m_3=2.25,\ m_4=2.75.$
2b	$B=\mathrm{f}(m_1)\times 0.5+\mathrm{f}(m_2)\times 0.5+\mathrm{f}(m_3)\times 0.5+\mathrm{f}(m_4)\times 0.5$ $=0.5\times\left[\mathrm{f}(1.25)+\mathrm{f}(1.75)+\mathrm{f}(2.25)+\mathrm{f}(2.75)\right]$ $=0.5\times\left[\mathrm{f}(1.25)+\mathrm{f}(1.25+1\times 0.5)+\mathrm{f}(1.25+2\times 0.5)+\mathrm{f}(1.25+3\times 0.5)\right]$ $=h\sum_{r=0}^{3}\mathrm{f}(1.25+rh)$ $a=m_1=1.25$.
2c	
2d	$\mathrm{f}(x)=\dfrac{1}{x}+1$  $B\approx 3.09$ (3 s.f.) $C\approx 3.28$ (3 s.f.) Actual area of $A=\int_1^3 \dfrac{1}{x}+1\,\mathrm{d}x$ ≈ 3.10 (3 s.f.) • The value of B is closer to the actual area of A • Comparing Figures 1 & 2, the gaps between the curve and the top of Irene's rectangles are larger than those of Yvonne's. ***(or other equivalent explanations)*** Thus Yvonne's estimation of the actual area of A is better.

3(a)	
3(b)	**<u>Method 1</u>** $\lvert x(x-5)\rvert=\sqrt{2}\lvert x-5\rvert$ $x(x-5)=\pm\sqrt{2}(x-5)$ $(x-5)(x\pm\sqrt{2})=0$ $x=5,\ \sqrt{2}\ or\ -\sqrt{2}$ **<u>Method 2</u>** $\lvert x(x-5)\rvert=\sqrt{2}\lvert x-5\rvert$ $x^2(x-5)^2=2(x-5)^2$ $(x-5)^2(x^2-2)=0$ $x-5=0\quad or\quad x^2=2$ $x=5,\ \sqrt{2}\ or\ -\sqrt{2}$ Thus at point $A, x=-\sqrt{2}$ and at point $B,\ x=\sqrt{2}$ From the graph, for $\lvert x(x-5)\rvert>\sqrt{2}\lvert x-5\rvert$, $x<-\sqrt{2}$ or $\sqrt{2}<x<5$ or $x>5$ **<u>Otherwise method:</u>** **<u>Solving inequality algebraically,</u>**

	$\lvert x(x-5)\rvert > \sqrt{2}\lvert x-5\rvert$ $x^2(x-5)^2 > 2(x-5)^2$ $(x-5)^2(x^2-2) > 0$ $(x-5)^2(x-\sqrt{2})(x+\sqrt{2}) > 0$ $\begin{array}{ccccccc} + & & - & & + & & + \\ \hline & -\sqrt{2} & & \sqrt{2} & & 5 & \end{array}$ $x < -\sqrt{2}$ or $\sqrt{2} < x < 5$ or $x > 5$

4 **(a)**	$\underset{\sim}{r} \times \underset{\sim}{q} = -\underset{\sim}{q} \times \underset{\sim}{p}$ $\underset{\sim}{r} \times \underset{\sim}{q} + \underset{\sim}{q} \times \underset{\sim}{p} = \underset{\sim}{0}$ $\underset{\sim}{r} \times \underset{\sim}{q} - \underset{\sim}{p} \times \underset{\sim}{q} = \underset{\sim}{0}$ $(\underset{\sim}{r} - \underset{\sim}{p}) \times \underset{\sim}{q} = \underset{\sim}{0}$ Since $\underset{\sim}{q} \neq \underset{\sim}{0}$, Either $(\underset{\sim}{r} - \underset{\sim}{p}) = \underset{\sim}{0}$ $\quad\cdots\cdots(1)$ or $(\underset{\sim}{r} - \underset{\sim}{p}) // \underset{\sim}{q} \Rightarrow (\underset{\sim}{r} - \underset{\sim}{p}) = k\underset{\sim}{q}$, $k \in \mathbb{R} \setminus \{0\}$ $\quad\cdots\cdots(2)$ Combining (1) & (2), $\Rightarrow \underset{\sim}{r} = \underset{\sim}{p} + k\underset{\sim}{q}$, $k \in \mathbb{R}$ The set of all possible positions of the point *R* form the line (OR *R* is any point on the line) passing through *P* and parallel to the vector $\underset{\sim}{q}$.
(b)	$\underset{\sim}{q} \bullet (\underset{\sim}{p} - \underset{\sim}{r}) = 0$ $\underset{\sim}{q} \bullet \underset{\sim}{p} - \underset{\sim}{q} \bullet \underset{\sim}{r} = 0$ $\underset{\sim}{r} \bullet \underset{\sim}{q} = \underset{\sim}{p} \bullet \underset{\sim}{q}$ $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \bullet \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} \bullet \begin{pmatrix} q_1 \\ q_2 \\ q_3 \end{pmatrix}$ $q_1 x + q_2 y + q_3 z = q_1 - 2q_2 + 3q_3$ The set of all possible positions of the point *R* form the plane (OR *R* is any point on the plane) that contains the point *P*(1, -2, 3) with a normal vector **q** (or perpendicular to **q)**

(c)	$\left\lvert \mathbf{q} \cdot (\mathbf{p}-\mathbf{c}) \right\rvert$ is the shortest distance from point C to the plane in (b). **OR** It is the length of projection of $\overrightarrow{CP}$ onto $\mathbf{q}$.

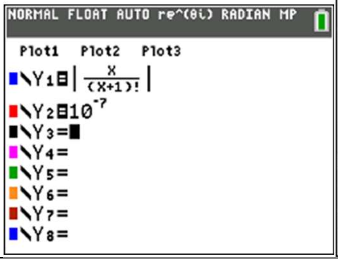
5(a)	$\frac{1}{(r-1)!}-\frac{2}{r!}+\frac{1}{(r+1)!}=\frac{r(r+1)}{(r+1)!}-\frac{2(r+1)}{(r+1)!}+\frac{1}{(r+1)!}$ $=\frac{r^2+r-2r-2+1}{(r+1)!}$ $=\frac{r^2-r-1}{(r+1)!}$ $\mathrm{f}(r)=r^2-r-1 \quad [\textit{Optional}]$

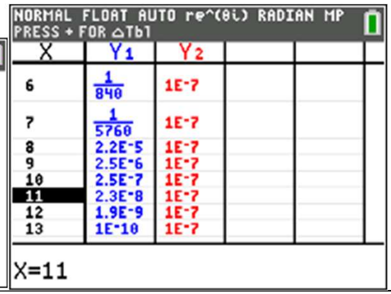
(b)	$\begin{aligned} \sum_{r=2}^{N}\frac{r^2-r-1}{(r+1)!} &= \sum_{r=2}^{N}\left(\frac{1}{(r-1)!}-\frac{2}{r!}+\frac{1}{(r+1)!}\right) \\ &= \left(\frac{1}{1!}-\frac{2}{2!}+\cancel{\frac{1}{3!}}\right) \\ &\quad +\left(\frac{1}{2!}-\cancel{\frac{2}{3!}}+\cancel{\frac{1}{4!}}\right) \\ &\quad +\left(\cancel{\frac{1}{3!}}-\cancel{\frac{2}{4!}}+\cancel{\frac{1}{5!}}\right) \\ &\quad +\left(\cancel{\frac{1}{4!}}-\cancel{\frac{2}{5!}}+\cancel{\frac{1}{6!}}\right) \\ &\quad \ldots \\ &\quad +\left(\cancel{\frac{1}{(N-3)!}}-\cancel{\frac{2}{(N-2)!}}+\cancel{\frac{1}{(N-1)!}}\right) \\ &\quad +\left(\cancel{\frac{1}{(N-2)!}}-\cancel{\frac{2}{(N-1)!}}+\frac{1}{N!}\right) \\ &\quad +\left(\cancel{\frac{1}{(N-1)!}}-\frac{2}{N!}+\frac{1}{(N+1)!}\right) \\ &= \frac{1}{2}+\frac{1}{N!}-\frac{2}{N!}+\frac{1}{(N+1)!} \\ &= \frac{1}{2}-\frac{1}{N!}+\frac{1}{(N+1)!} \\ &= \frac{1}{2}-\frac{N}{(N+1)!} \end{aligned}$
(c)	As $N\to\infty$, $-\frac{1}{N!}+\frac{1}{(N+1)!}\to 0$, hence $S_N\to\frac{1}{2}$, which is a finite value. Therefore S_N converges. **OR** As $N\to\infty$, $\frac{N}{(N+1)!}\to 0$, hence $S_N\to\frac{1}{2}$, which is a finite value. Therefore S_N converges. $S_\infty=\frac{1}{2}$
(d)	$S_\infty=\frac{1}{2}$

$$\left|S_\infty - S_N\right| = \left|\frac{1}{N!} - \frac{1}{(N+1)!}\right| < 10^{-7}$$

OR $\left|S_\infty - S_N\right| = \left|\frac{N}{(N+1)!}\right| < 10^{-7}$

Using GC, smallest $N = 11$





6 (a)

Method 1

$$\begin{aligned}
1+\mathrm{e}^{-\mathrm{i}\alpha} &= \mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}\left(\mathrm{e}^{\mathrm{i}\frac{\alpha}{2}} + \mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}\right) \\
&= \mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}\left[\cos\frac{\alpha}{2} + \mathrm{i}\sin\frac{\alpha}{2} + \cos\left(-\frac{\alpha}{2}\right) + \mathrm{i}\sin\left(-\frac{\alpha}{2}\right)\right] \\
&= \mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}\left[\cos\frac{\alpha}{2} + \mathrm{i}\sin\frac{\alpha}{2} + \cos\frac{\alpha}{2} - \mathrm{i}\sin\frac{\alpha}{2}\right] \\
&= 2\cos\frac{\alpha}{2}\,\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}
\end{aligned}$$

Method 2

$$\begin{aligned}
1+\mathrm{e}^{-\mathrm{i}\alpha} &= 1 + \cos(-\alpha) + \mathrm{i}\sin(-\alpha) \\
&= 1 + \cos\alpha - \mathrm{i}\sin\alpha \\
&= 1 + \left(2\cos^2\frac{\alpha}{2} - 1\right) - 2\mathrm{i}\sin\frac{\alpha}{2}\cos\frac{\alpha}{2} \\
&= 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} - \mathrm{i}\sin\frac{\alpha}{2}\right) \\
&= 2\cos\frac{\alpha}{2}\left(\cos\left(-\frac{\alpha}{2}\right) + \mathrm{i}\sin\left(-\frac{\alpha}{2}\right)\right) \\
&= 2\cos\frac{\alpha}{2}\,\mathrm{e}^{-\mathrm{i}\frac{\alpha}{2}}
\end{aligned}$$

(b)	$\left(1+e^{-i\alpha}\right)^3-\left(1+e^{i\alpha}\right)^3$ $=\left(2\cos\frac{\alpha}{2}e^{-i\frac{\alpha}{2}}\right)^3-\left(2\cos\left(-\frac{\alpha}{2}\right)e^{-i\left(-\frac{\alpha}{2}\right)}\right)^3$ $=8\cos^3\frac{\alpha}{2}e^{-i\frac{3\alpha}{2}}-8\cos^3\frac{\alpha}{2}e^{i\frac{3\alpha}{2}}$ $=8\cos^3\frac{\alpha}{2}\left(e^{-i\frac{3\alpha}{2}}-e^{i\frac{3\alpha}{2}}\right)$ $=8\cos^3\frac{\alpha}{2}\left\{\left[\cos\left(-\frac{3\alpha}{2}\right)+i\sin\left(-\frac{3\alpha}{2}\right)\right]-\left[\cos\left(\frac{3\alpha}{2}\right)+i\sin\left(\frac{3\alpha}{2}\right)\right]\right\}$ $=8\cos^3\frac{\alpha}{2}\left[\cos\left(\frac{3\alpha}{2}\right)-i\sin\left(\frac{3\alpha}{2}\right)-\cos\left(\frac{3\alpha}{2}\right)-i\sin\left(\frac{3\alpha}{2}\right)\right]$ $=8\cos^3\frac{\alpha}{2}\left(-2i\sin\frac{3\alpha}{2}\right)$ $=-16i\cos^3(\frac{\alpha}{2})\sin\left(\frac{3\alpha}{2}\right)$
(c)	Given that $0<\alpha<\frac{2}{3}\pi$, $0<\frac{\alpha}{2}<\frac{\pi}{3} \quad \Rightarrow \cos\frac{\alpha}{2}>0$ $0<\frac{3}{2}\alpha<\pi \quad \Rightarrow \sin\frac{3}{2}\alpha>0$ $z=-16i\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{3\alpha}{2}\right)$ $=16\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{3\alpha}{2}\right)e^{-i\frac{\pi}{2}}$ $\lvert z\rvert=16\cos^3\left(\frac{\alpha}{2}\right)\sin\left(\frac{3\alpha}{2}\right);\quad \arg(z)=-\frac{\pi}{2}$

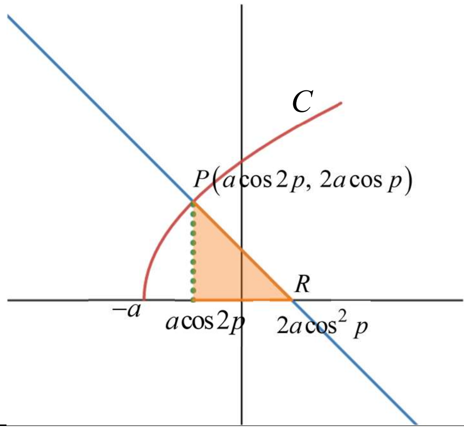
7 (a)	$\mathrm{f}(x)=\ln(1+\sin 2x)+2$ **<u>Method 1</u>** $\mathrm{f}'(x)=\dfrac{2\cos 2x}{1+\sin 2x}$ $(1+\sin 2x)\mathrm{f}'(x)=2\cos 2x$ $(1+\sin 2x)\mathrm{f}''(x)+(2\cos 2x)\mathrm{f}'(x)=-4\sin 2x$ ---- (*) $\mathrm{f}'(0)=\dfrac{2\cos 0}{1+\sin 0}=2$, $(1+\sin 0)\mathrm{f}''(0)+(2\cos 0)(2)=-4\sin 0$ $\mathrm{f}''(0)=-4$ $\mathrm{f}(0)=\ln(1+\sin 0)+2=2$ $\therefore\ \mathrm{f}(x)=2+2x-2x^2+\dots$ **<u>Method 2</u>** $\mathrm{f}'(x)=\dfrac{2\cos 2x}{1+\sin 2x}$ $\mathrm{f}''(x)=\dfrac{(1+\sin 2x)(-4\sin 2x)-(2\cos 2x)(2\cos 2x)}{(1+\sin 2x)^2}$ $=\dfrac{-4\sin 2x-4\sin^2 2x-4\cos^2 2x}{(1+\sin 2x)^2}$ $=\dfrac{-4(1+\sin 2x)}{(1+\sin 2x)^2}$ $=\dfrac{-4}{(1+\sin 2x)}$ $\mathrm{f}'(0)=\dfrac{2\cos 0}{1+\sin 0}=2$, $\mathrm{f}''(0)=-4$ $\mathrm{f}(0)=\ln(1+\sin 0)+2=2$ $\therefore\ \mathrm{f}(x)=2+2x-2x^2+\dots$

	Method 3 $$y=\ln(1+\sin 2x)+2$$ $$e^{y-2}=1+\sin 2x$$ $$e^{y-2}\frac{dy}{dx}=2\cos 2x$$ $$e^{y-2}\frac{d^2y}{dx^2}+e^{y-2}\left(\frac{dy}{dx}\right)^2=-4\sin 2x$$ $e^0\frac{dy}{dx}=2 \Rightarrow f'(0)=2$ $\frac{d^2y}{dx^2}+2^2=0 \Rightarrow f''(0)=-4$, $f(0)=\ln(1+\sin 0)+2=2$ $\therefore f(x)=2+2x-2x^2+\ldots$
(b)	$\frac{1}{\cos 2x+\sin x}\approx\frac{1}{1-\frac{(2x)^2}{2}+x}$ $=\left[1+(x-2x^2)\right]^{-1}$ $=1-(x-2x^2)+\frac{(-1)(-2)}{2!}(x-2x^2)^2+\ldots$ $=1-x+2x^2+x^2+\ldots$ $=1-x+3x^2+\ldots$

8(a)	$\tan\theta$ $=\tan(\angle BAD-\angle BAC)$ $=\frac{\tan\angle BAD-\tan\angle BAC}{1+(\tan\angle BAD)(\tan\angle BAC)}$ $=\frac{\frac{a+3}{x}-\frac{a}{x}}{1+\frac{a+3}{x}\cdot\frac{a}{x}}$ $=\frac{\frac{3}{x}}{1+\frac{a(a+3)}{x^2}}\times\frac{x^2}{x^2}$ $=\frac{3x}{x^2+3a+a^2}$

(b)	Let $y = \tan\ \theta = \dfrac{3x}{x^2+3a+a^2}$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\left(x^2+3a+a^2\right)3 - 3x\left(2x\right)}{\left(x^2+3a+a^2\right)^2}$ $= \dfrac{-3x^2+3a^2+9a}{\left(x^2+3a+a^2\right)^2}$ At stationary value of tan θ, $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$ $\dfrac{-3x^2+3a^2+9a}{\left(x^2+3a+a^2\right)^2} = 0$ $-3x^2+3a^2+9a = 0$ $3x^2 = 9a+3a^2$ $x = \sqrt{a\left(3+a\right)}$ (reject $-\sqrt{a\left(3+a\right)}$ as $x > 0$) $\tan\theta = \dfrac{3\sqrt{a(3+a)}}{a(3+a)+3a+a^2}$ $= \dfrac{3\sqrt{a(3+a)}}{a(3+a)+a(3+a)}$ $= \dfrac{3\sqrt{a(3+a)}}{2a(a+3)}$ *or* $\dfrac{3}{2\sqrt{a(a+3)}}$
(c)	$\tan\angle ADB = \dfrac{x}{a+3}$ $= \dfrac{\sqrt{a\left(a+3\right)}}{a+3}$ $= \sqrt{\dfrac{a}{a+3}}$ $\tan\angle ADB = \sqrt{\dfrac{\frac{a}{a}}{\frac{a+3}{a}}} = \sqrt{\dfrac{1}{1+\dfrac{3}{a}}}$ Since a is much greater than 3, then $\dfrac{3}{a} \approx 0$ $\tan\angle ADB \approx 1$. Thus $\angle ADB \approx 45^\circ$.

9 (a)	$x = a\cos 2t, \qquad y = 2a\cos t$ $\dfrac{\mathrm{d}x}{\mathrm{d}t} = -2a\sin 2t, \qquad \dfrac{\mathrm{d}y}{\mathrm{d}t} = -2a\sin t$ $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\sin t}{\sin 2t}$ $= \dfrac{\sin t}{2\sin t\cos t}$ $= \dfrac{1}{2\cos t}$ At P, $t = p$. Gradient of normal at $P = -2\cos p$ Equation of normal at P is $y - 2a\cos p = -2\cos p\left(x - a\cos 2p\right)$ $y = -2\cos p\left[x - a\left(2\cos^2 p - 1\right)\right] + 2a\cos p$ $y = -2\cos p\left(x + a - 2a\cos^2 p\right) + 2a\cos p$ $y = -2\cos p\left(x - 2a\cos^2 p\right)$
9 (b)	At R, $y = 0$ $-2\cos p\left(x - 2a\cos^2 p\right) = 0$ $x = 2a\cos^2 p$ When C meets the x-axis, $y = 0 \Rightarrow 2a\cos t = 0$ $t = \dfrac{\pi}{2}$ $x = a\cos\pi$ $= -a$ Required area $= \displaystyle\int_{-a}^{a\cos 2p} y \,\mathrm{d}x + \frac{1}{2}\left[2a\cos^2 p - a\cos 2p\right]\left(2a\cos p\right)$ $= \displaystyle\int_{\frac{\pi}{2}}^{p} 2a\cos t\left(-2a\sin 2t\right)\,\mathrm{d}t + \left(a^2\cos p\right)\left[2\cos^2 p - \left(2\cos^2 p\right.\right.$ $= 4a^2\displaystyle\int_{p}^{\frac{\pi}{2}} \cos t\left(\sin 2t\right)\,\mathrm{d}t + a^2\cos p$ (Shown)

	 *Diagram is not required. For reference only.*
9 (c)	**Method 1: (Using double angle formula for sin2t)**

Area

$$= 4a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \cos t\left(2\sin t\cos t\right)\,\mathrm{d}t \;+\; a^2\cos\frac{\pi}{3}$$

$$= -8a^2\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \left(-\sin t\right)\left(\cos^2 t\right)\,\mathrm{d}t \;+\; \frac{1}{2}a^2$$

$$= -8a^2\left[\frac{1}{3}\cos^3 t\right]_{\frac{\pi}{3}}^{\frac{\pi}{2}} \;+\; \frac{1}{2}a^2$$

$$= -\frac{8}{3}a^2\left[0-\cos^3\frac{\pi}{3}\right] \;+\; \frac{1}{2}a^2$$

$$= \frac{1}{3}a^2+\frac{1}{2}a^2$$

$$= \frac{5}{6}a^2$$

	Method 2: (Using factor formula for cos*t* sin2*t* $= 4a^2 \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{2}(\sin 3t + \sin t)\ \mathrm{d}t\ +\ a^2 \cos\frac{\pi}{3}$ $= 2a^2 \left[-\frac{1}{3}\cos 3t - \cos t \right]_{\frac{\pi}{3}}^{\frac{\pi}{2}}\ +\ \frac{1}{2}a^2$ $= 2a^2 \left[0 + \frac{1}{3}\cos\pi + \cos\frac{\pi}{3} \right] + \frac{1}{2}a^2$ $= 2a^2 \left(-\frac{1}{3} + \frac{1}{2} \right)\ +\ \frac{1}{2}a^2$ $= \frac{1}{3}a^2 + \frac{1}{2}a^2$ $= \frac{5}{6}a^2$

10 (a)	85% of 450000 = 382 500 The amount owed at the end of July 2022 is (1+ 0.002) ×(382500 – x) = 1.002(382500 – x)
(b)	(table below)

No. of repay-ment	Amount of money owed after each repayment (beginning of month)	At the end of the month after adding interest
1	$a - x$, where $a = 382500$,	$1.002(a - x)$
2	$1.002(a - x) - x$ $= 1.002a - 1.002x - x$	$1.002(1.002a - 1.002x - x)$
3	$1.002(1.002a - 1.002x - x)$ $-x$ $= 1.002^2 a - 1.002^2 x - 1.002x$ $-x$	$1.002(1.002^2 a - 1.002^2 x - 1.002x - x)$
…		
n	$1.002^{n-1} a - 1.002^{n-1} x$ $-1.002^{n-2} x - ... - x$	

	Amount of money owed after the n^{th} repayment at the beginning of the month $=1.002^{n-1}a-1.002^{n-1}x-1.002^{n-2}x-...-x$ --- (*) $=1.002^{n-1}a-x\left(1+1.002+1.002^2+...+1.002^{n-2}+1.002^{n-1}\right)$ $=1.002^{n-1}a-x\left(\dfrac{1.002^n-1}{1.002-1}\right)$ $=1.002^{n-1}(382500)-x\left(\dfrac{1.002^n-1}{0.002}\right)$ $=1.002^{n-1}(382500)-500x(1.002^n-1)$ (Shown)
(c)	When $x = 2000$, $1.002^{n-1}(382500)-500(2000)(1.002^n-1)\le 0$ From GC, $n\ge 240.66$ He will pay off his loan on the 241 repayments, that is, 20 years 1 month. So the earliest date is 1 July 2042 after he makes the 241th repayments. At the end of 240th month after interest is added, amount owed is $1.002\times\left(1.002^{239}a-1.002^{239}x-1.002^{238}x-...-x\right)$ $1.002^{240}\times 382500-2000\left[1.002^{240}+1.002^{239}+\cdots+1.002\right]$ $=1.002^{240}\times 382500-2000\times 1.002\left(\dfrac{1.002^{240}-1}{1.002-1}\right)$ $=1321.71$ (2 d.p.) The amount of repayment on 1 July 2042 is \$1321.71
(d)	From 1 July 2022 to 1 Jan 2050, he would have made 27× 12 + 7 = 331 repayments. $1.002^{331-1}(382500)-500x(1.002^{331}-1)=0$ $x=\dfrac{1.002^{331-1}(382500)}{500(1.002^{331}-1)}\approx 1577.94$

11(a)	$\dfrac{\mathrm{d}N}{\mathrm{d}t}=k\left(N-120\right),\quad k>0$

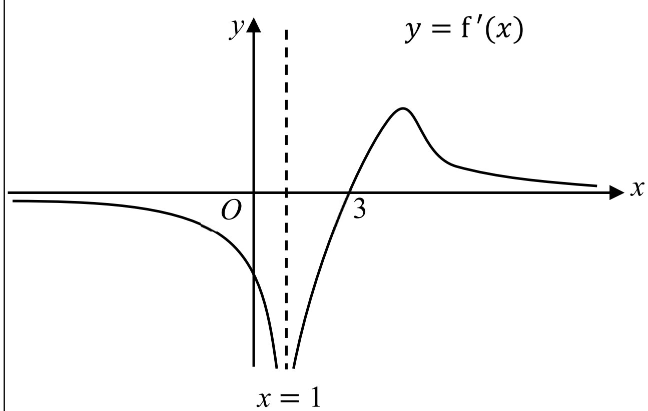
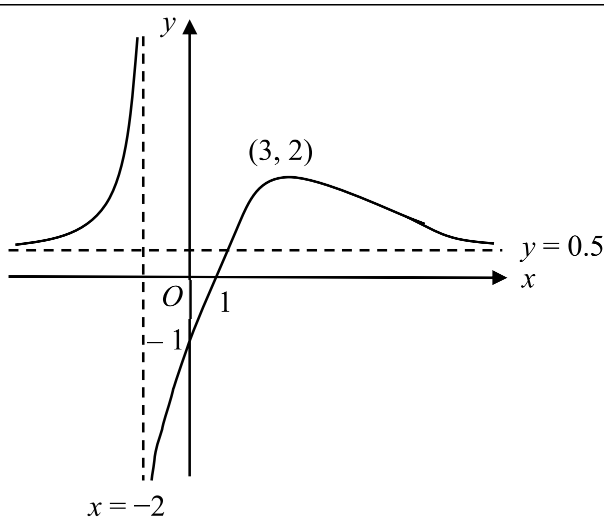
11(b)	$\int \frac{1}{N-120}\,\mathrm{d}N = \int k\,\mathrm{d}t$ $\ln\lvert N-120\rvert = kt + C$ $N-120 = \pm \mathrm{e}^{kt+C}$ $N = 120 + A\mathrm{e}^{kt}$ where $A = \pm \mathrm{e}^{C}$ When $t=0,\ N=600 \Rightarrow A = 600-120$ $= 480$ When $N=750,\ \frac{\mathrm{d}N}{\mathrm{d}t} = 63 \Rightarrow 63 = k(750-120)$ $k = \frac{1}{10}$ Thus $N = 120 + 480\mathrm{e}^{\frac{1}{10}t}$ *or* $120\left(1+4\mathrm{e}^{\frac{1}{10}t}\right)$
11(c)	**<u>Method 1</u>** When $t=15,\ N = 120+480\mathrm{e}^{1.5} = 2271.21 < 2500$ The target will not be met in 15 years. **<u>Method 2</u>** When $N=2500,\ \ 120+480\mathrm{e}^{\frac{1}{10}t} = 2500$ $t = 16.01 > 15$ The target will not be met in 15 years.
11(d)	$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{2}\sqrt{\left(24-\frac{1}{3}h\right)}$ When h is maximum, $\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{2}\sqrt{\left(24-\frac{1}{3}h\right)} = 0$ $h = 24\times 3 = 72$ The maximum height of the plant is 72 cm.

11(e)	$\int \left(24-\frac{1}{3}h\right)^{-1/2} \mathrm{d}h = \int \frac{1}{2}\mathrm{d}t$ $\frac{\left(24-\frac{1}{3}h\right)^{1/2}}{\left(\frac{1}{2}\right)\left(-\frac{1}{3}\right)} = \frac{1}{2}t + C$ $\frac{1}{2}t + C = -6\sqrt{\left(24-\frac{1}{3}h\right)}$ When $t=0,\ h=0,\ C=-6\sqrt{24}=-12\sqrt{6}$ $t = 24\sqrt{6}-12\sqrt{\left(24-\frac{1}{3}h\right)}$ When $h=24,\ t=24\sqrt{6}-12\sqrt{(24-8)}$ $=10.8$ (3 s.f.) It takes 10.8 years to reach a height of 24 cm.

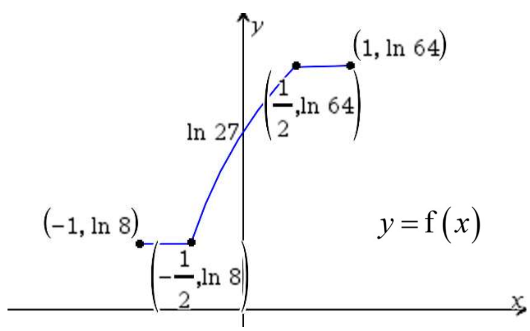
2022 JC2 H2MA Preliminary Examination Paper 2

1(a)	Let $u = x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$ $\frac{\mathrm{d}v}{\mathrm{d}x} = (4-2x)^{-\frac{1}{2}} \Rightarrow v = -(4-2x)^{\frac{1}{2}}$ $I = \int \frac{x}{\sqrt{4-2x}}\ \mathrm{d}x$ $= -x(4-2x)^{\frac{1}{2}} + \int (4-2x)^{\frac{1}{2}}\ \mathrm{d}x$ $= -x(4-2x)^{\frac{1}{2}} + \frac{(4-2x)^{\frac{3}{2}}}{\frac{3}{2}(-2)} + C$ $= -x(4-2x)^{\frac{1}{2}} - \frac{(4-2x)^{\frac{3}{2}}}{3} + C$
(b)	$u = 4-2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = -2$ $I = \int \frac{x}{\sqrt{4-2x}}\ \mathrm{d}x$ $= \int \frac{4-u}{2\sqrt{u}} \left(-\frac{1}{2}\right) \mathrm{d}u$ $= \int \frac{u-4}{4\sqrt{u}}\ \mathrm{d}u$ $= \int \frac{1}{4}u^{\frac{1}{2}} - u^{-\frac{1}{2}}\ \mathrm{d}u$ $= \frac{1}{6}u^{\frac{3}{2}} - 2u^{\frac{1}{2}} + D$ $= \frac{1}{6}(4-2x)^{\frac{3}{2}} - 2(4-2x)^{\frac{1}{2}} + D$

(c)	$\left[-x(4-2x)^{\frac{1}{2}}-\frac{(4-2x)^{\frac{3}{2}}}{3}+C\right]-\left[\frac{1}{6}(4-2x)^{\frac{3}{2}}-2(4-2x)^{\frac{1}{2}}+D\right]$ $=-x(4-2x)^{\frac{1}{2}}-\frac{(4-2x)^{\frac{3}{2}}}{3}-\frac{1}{6}(4-2x)^{\frac{3}{2}}+2(4-2x)^{\frac{1}{2}}+C-D$ $=(4-2x)^{\frac{1}{2}}\left[-x-\frac{1}{3}(4-2x)-\frac{1}{6}(4-2x)+2\right]+C-D$ $=(4-2x)^{\frac{1}{2}}\left(-x+\frac{2}{3}x+\frac{1}{3}x-\frac{4}{3}-\frac{2}{3}+2\right)+C-D$ $=(4-2x)^{\frac{1}{2}}(0)+C-D$ $=C-D$ (shown)

Qn	Solution
2(a)	
2(b)	

3(a)	Largest possible $k = -1.5$ $y = x^2 + 3x - 1$ $= (x+1.5)^2 - 2.25 - 1$ $(x+1.5)^2 = y + 3.25$ $x = -1.5 \pm \sqrt{y+3.25}$ since $x \le -1.5$ $x = -1.5 - \sqrt{y+3.25}$ $f^{-1}(x) = -1.5 - \sqrt{x+3.25}$
(b)	$f(x) = f^{-1}(x)$ Since the graphs of f and f^{-1} intersect at $y = x$, $f(x) = x$ $x^2 + 3x - 1 = x$ $x^2 + 2x - 1 = 0$ $x = \frac{-2 \pm \sqrt{2^2 - 4(-1)}}{2}$ $= \frac{-2 \pm \sqrt{8}}{2}$ $x = -1 - \sqrt{2}$ or $x = -1 + \sqrt{2}$ Since $x \le -2$, $\therefore\ x = -1 - \sqrt{2}$
(c)	fg does not exist as $R_g = [0, \infty) \not\subseteq D_f = (-\infty, -2]$ gf exists as $R_f = [-3, \infty) \subseteq D_g = [-5, \infty)$ $gf(x) = g(x^2 + 3x - 1)$ $= \sqrt{x^2 + 3x - 1 + 5}$ $= \sqrt{x^2 + 3x + 4},\ x \le -2$
(d)	$R_f = [-3, \infty)$ $R_{gf} = [\sqrt{2}, \infty)$

	Solution
4(a)	$\ln\left(2x+3\right)^3 = 3\ln(2x+3) = 3\text{g}(2x+3)$ where $\text{g}(x) = \ln x$. This is the sequence of transformations: (1) Translate the graph $y = \ln x$ 3 units in the negative x-direction (2) Scale parallel to x-axis by a scale factor of ½ (3) Scale parallel to y-axis by a scale factor of 3
(b)	
(c)	$y = \ln\left(2x+3\right)^3$ $e^y = (2x+3)^3$ $x = \frac{1}{2}\left(\left(e^y\right)^{\frac{1}{3}} - 3\right)$ $x^2 = \frac{1}{4}\left(e^{\frac{y}{3}} - 3\right)^2$ Required volume = volume of cylinder, with radius 1, height (ln 64 – ln 27) $-\pi\int_{\ln 27}^{\ln 64} x^2 \,\text{d}y$ $= \pi(1^2)(\ln 64 - \ln 27) - \pi\int_{\ln 27}^{\ln 64} \frac{1}{4}\left(e^{\frac{y}{3}} - 3\right)^2 \,\text{d}y$ = 2.501 (to 3 d.p.)

5(a)

Method 1

$$\overrightarrow{AB} = \begin{pmatrix}0\\4\\2\end{pmatrix} - \begin{pmatrix}4\\2\\-1\end{pmatrix} = \begin{pmatrix}-4\\2\\3\end{pmatrix}$$

$$l\text{: } \mathbf{r} = \begin{pmatrix}0\\4\\2\end{pmatrix} + t\begin{pmatrix}-4\\2\\3\end{pmatrix}, t \in \mathbb{R} \quad \text{or} \quad \mathbf{r} = \begin{pmatrix}4\\-2\\1\end{pmatrix} + t\begin{pmatrix}-4\\2\\3\end{pmatrix}, t \in \mathbb{R}$$

$$\text{Plane } p\text{: } \mathbf{r} = \begin{pmatrix}6\\2\\-5\end{pmatrix} + \lambda\begin{pmatrix}0\\3\\-1\end{pmatrix} + \mu\begin{pmatrix}2\\-2\\1\end{pmatrix}$$

Normal of plane p:

$$\begin{pmatrix}0\\3\\-1\end{pmatrix} \times \begin{pmatrix}2\\-2\\1\end{pmatrix} = \begin{pmatrix}3(1)-(-2)(-1)\\-(0(1)-2(-1))\\0(-2)-2(3)\end{pmatrix} = \begin{pmatrix}1\\-2\\-6\end{pmatrix}$$

$$\text{Plane } p\text{: } \mathbf{r} \cdot \begin{pmatrix}1\\-2\\-6\end{pmatrix} = \begin{pmatrix}6\\2\\-5\end{pmatrix} \cdot \begin{pmatrix}1\\-2\\-6\end{pmatrix} = 32$$

Let X be the point of intersection between plane p and line l.

$$\overrightarrow{OX} = \begin{pmatrix}0\\4\\2\end{pmatrix} + t\begin{pmatrix}-4\\2\\3\end{pmatrix}, \text{for some } t \in \mathbb{R}$$

$$\text{Or } \overrightarrow{OX} = \begin{pmatrix}4\\2\\-1\end{pmatrix} + t\begin{pmatrix}-4\\2\\3\end{pmatrix}, \text{for some } t \in \mathbb{R}$$

Since X lies on p,

$$\left[\begin{pmatrix}0\\4\\2\end{pmatrix} + t\begin{pmatrix}-4\\2\\3\end{pmatrix}\right] \cdot \begin{pmatrix}1\\-2\\-6\end{pmatrix} = 32$$

$$(-8-12) + t(-4-4-18) = 32$$

$$-26t = 52$$

$$t = -2$$

or

$$\left[\begin{pmatrix}4\\2\\-1\end{pmatrix} + t\begin{pmatrix}-4\\2\\3\end{pmatrix}\right] \cdot \begin{pmatrix}1\\-2\\-6\end{pmatrix} = 32$$

$$(4-4+6) + t(-4-4-18) = 32$$

$$-26t = 26$$

$$t = -1$$

$$\overrightarrow{OX} = \begin{pmatrix}0\\4\\2\end{pmatrix} - 2\begin{pmatrix}-4\\2\\3\end{pmatrix} = \begin{pmatrix}8\\0\\-4\end{pmatrix} \quad \text{or} \quad \overrightarrow{OX} = \begin{pmatrix}4\\2\\-1\end{pmatrix} - \begin{pmatrix}-4\\2\\3\end{pmatrix} = \begin{pmatrix}8\\0\\-4\end{pmatrix}$$

The coordinates of the point of intersection is $(8,0,-4)$

Method 2

	Equating $\mathbf{r}=\begin{pmatrix}0\\4\\2\end{pmatrix}+t\begin{pmatrix}-4\\2\\3\end{pmatrix}, t\in\mathbb{R}$ and $\mathbf{r}=\begin{pmatrix}6\\2\\-5\end{pmatrix}+\lambda\begin{pmatrix}0\\3\\-1\end{pmatrix}+\mu\begin{pmatrix}2\\-2\\1\end{pmatrix}$, $\begin{pmatrix}0\\4\\2\end{pmatrix}+t\begin{pmatrix}-4\\2\\3\end{pmatrix}=\begin{pmatrix}6\\2\\-5\end{pmatrix}+\lambda\begin{pmatrix}0\\3\\-1\end{pmatrix}+\mu\begin{pmatrix}2\\-2\\1\end{pmatrix}$ $-4t=6+0\lambda+2\mu \Rightarrow 0\lambda+2\mu+4t=-6$ ---- (1) $4+2t=2+3\lambda-2\mu \Rightarrow 3\lambda-2\mu-2t=2$ ---- (2) $2+3t=-5-\lambda+\mu \Rightarrow -\lambda+\mu-3t=7$ ---- (3) Using GC, $\lambda=0,\ \mu=1,\ t=-2$ When $t=-2$, $\overrightarrow{OX}=\begin{pmatrix}0\\4\\2\end{pmatrix}-2\begin{pmatrix}-4\\2\\3\end{pmatrix}=\begin{pmatrix}8\\0\\-4\end{pmatrix}$ The coordinates of the point of intersection is $(8,0,-4)$.
(b)	Let the equation of either plane be $\mathbf{r}\bullet\begin{pmatrix}1\\-2\\-6\end{pmatrix}=D$. Distance between either plane and plane $p=\sqrt{41}$ $\Rightarrow \left\lvert \dfrac{D-32}{\sqrt{1^2+2^2+6^2}} \right\rvert=\sqrt{41}$ $\Rightarrow \left\lvert \dfrac{D-32}{\sqrt{41}} \right\rvert=\sqrt{41}$ $\Rightarrow\ D-32=41$ or $D-32=-41$ $\Rightarrow\ D=73$ or $D=-9$ $\therefore$ the equations of the planes are $\mathbf{r}\bullet\begin{pmatrix}1\\-2\\-6\end{pmatrix}=73$ or $\mathbf{r}\bullet\begin{pmatrix}1\\-2\\-6\end{pmatrix}=-9$ $x-2y-6z=73$ or $x-2y-6z=-9$

(c)	$m:\mathbf{r}=\begin{pmatrix}1\\-2\\3\end{pmatrix}+t\begin{pmatrix}-3\\1\\a\end{pmatrix}, t\in\mathbb{R}$ Since p and m do not meet in a unique point, p and m are parallel. $\begin{pmatrix}-3\\1\\a\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\-6\end{pmatrix}=0$ $-3-2-6a=0$ $a=-\frac{5}{6}$

6(a)	**<u>Method 1:</u>** Case 1: 3W2M – No. of ways $={}^{7}C_{3}\,{}^{5}C_{2}=350$ Case 2: 4W1M – No. of ways $={}^{7}C_{4}\,{}^{5}C_{1}=175$ Case 3: 5W0M – No. of ways $={}^{7}C_{5}\,{}^{5}C_{0}=21$ Total number of ways $=350+175+21=546$ **<u>Method 2:</u>** (Using complementary method) Case 1: 0W5M – No. of ways $={}^{7}C_{0}\,{}^{5}C_{5}=1$ Case 2: 1W4M – No. of ways $={}^{7}C_{1}\,{}^{5}C_{4}=35$ Case 2: 2W3M – No. of ways $={}^{7}C_{2}\,{}^{5}C_{3}=210$ Total number of ways $={}^{12}C_{5}-1-35-210=546$
6(b)	Required probability $=\frac{(7-1)!\times 2!\ \times {}^{6}C_{4}\times 4!}{(12-1)!}$ $=\frac{1}{77}$

7(a)	$\mathrm{P}(X)$ $=\mathrm{P}(\text{Two 3s or two 9s and one other})$ $=\mathrm{P}(A3, B\text{ any}, C3)+\big[\mathrm{P}(A9, B9, C\text{ not}9)$ $+\mathrm{P}(A9, B\text{ not}9, C9)+\mathrm{P}(A\text{ not}9, B9, C9)\big]$ $=\left(\frac{1}{4}\right)\left(\frac{5}{5}\right)\left(\frac{1}{8}\right)+\left[\left(\frac{1}{4}\right)\left(\frac{2}{5}\right)\left(\frac{5}{8}\right)+\left(\frac{1}{4}\right)\left(\frac{3}{5}\right)\left(\frac{3}{8}\right)+\left(\frac{3}{4}\right)\left(\frac{2}{5}\right)\left(\frac{3}{8}\right)\right]$ $=\frac{21}{80}$
7(b)	$\mathrm{P}(X\cap Y)=\mathrm{P}(A5, B9, C9)$ $=\left(\frac{1}{4}\right)\left(\frac{2}{5}\right)\left(\frac{3}{8}\right)$ $=\frac{3}{80}$ $\mathrm{P}(Y)=\mathrm{P}(A5, B\text{ any}, C\text{ any})=\frac{1}{4}$ $\mathrm{P}(X)\times\mathrm{P}(Y)=\frac{21}{320}\neq\mathrm{P}(X\cap Y)$ Therefore, events X and Y are not independent.
7(c)	$\mathrm{P}(\text{One ball is numbered 7}\mid X)$ $=\dfrac{\mathrm{P}(A3, B7, C3)+\mathrm{P}(A9, B7, C9)}{\mathrm{P}(X)}$ $=\dfrac{\left(\frac{1}{4}\right)\left(\frac{1}{5}\right)\left(\frac{1}{8}\right)+\left(\frac{1}{4}\right)\left(\frac{1}{5}\right)\left(\frac{3}{8}\right)}{\frac{21}{80}}$ $=\frac{2}{21}$
8(a)	Possible values of T are $\{a, 2a, b, 3b\}$. Since $\mathrm{P}(T=2a)=\mathrm{P}(T=3b)=\frac{1}{6}$, then $\mathrm{P}(T=a)+\mathrm{P}(T=b)=\frac{4}{6}$. If $\mathrm{P}(T=a)=\mathrm{P}(T=b)=\frac{2}{6}$, then the modes of T are a and b, which contradicts the question. Hence, $\mathrm{P}(T=a)=\frac{3}{6}=\frac{1}{2}$ and $\mathrm{P}(T=b)=\frac{1}{6}$.

	t	a	$2a$	b	$3b$
	$\mathrm{P}(T=t)$	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$

8(b)

$$\mathrm{E}(T)=a\left(\frac{1}{2}\right)+2a\left(\frac{1}{6}\right)+b\left(\frac{1}{6}\right)+3b\left(\frac{1}{6}\right)$$
$$=\frac{5}{6}a+\frac{2}{3}b$$

Given that $\mathrm{E}(T)=\frac{25}{6}$,

$$\frac{5}{6}a+\frac{2}{3}b=\frac{25}{6}\Rightarrow b=\frac{25}{4}-\frac{5}{4}a$$
$$=\frac{5}{4}(5-a)$$

$$\mathrm{E}\left(T^2\right)=a^2\left(\frac{1}{2}\right)+(2a)^2\left(\frac{1}{6}\right)+b^2\left(\frac{1}{6}\right)+(3b)^2\left(\frac{1}{6}\right)$$
$$=\frac{1}{2}a^2+\frac{2}{3}a^2+\frac{1}{6}b^2+\frac{3}{2}b^2$$
$$=\frac{7}{6}a^2+\frac{5}{3}b^2$$

$$\mathrm{Var}(T)=\mathrm{E}\left(T^2\right)-\left[\mathrm{E}(T)\right]^2$$
$$=\frac{7}{6}a^2+\frac{5}{3}b^2-\left(\frac{25}{6}\right)^2$$
$$=\frac{7}{6}a^2+\frac{5}{3}\left[\frac{5}{4}(5-a)\right]^2-\frac{625}{36}$$
$$=\frac{7}{6}a^2+\frac{125}{48}(5-a)^2-\frac{625}{36}$$
$$=\frac{181}{48}a^2-\frac{625}{24}a+\frac{6875}{144}$$

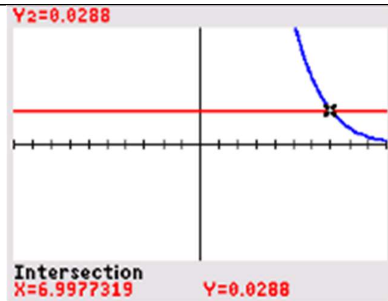
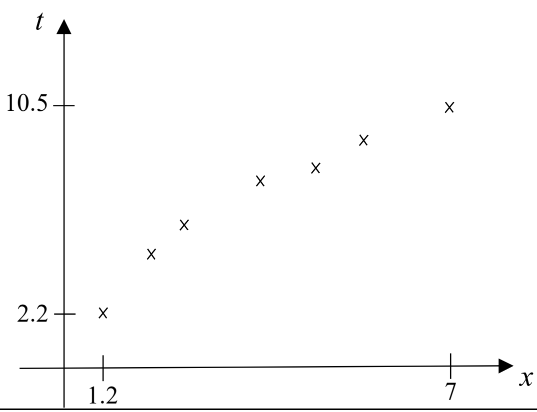
9(a)

$$X\sim B\left(75,\frac{p}{100}\right)$$

$\mathrm{P}(X\leq 1)=0.0288$

$$\binom{75}{0}\left(\frac{p}{100}\right)^0\left(1-\frac{p}{100}\right)^{75}+\binom{75}{1}\left(\frac{p}{100}\right)\left(1-\frac{p}{100}\right)^{74}=0.0288$$

$$\left(1-\frac{p}{100}\right)^{75}+\frac{3}{4}p\left(1-\frac{p}{100}\right)^{74}=0.0288$$

	 p =6.9977 (to 4 d.p.)
9(b)	$X \sim B(75, 0.075)$ From GC, $\mathrm{P}(X=4)=0.1517$ $\mathrm{P}(X=5)=0.1747$ $\mathrm{P}(X=6)=0.1652$ The most likely number of bouquets with LED lights made in a day is 5.
9(c)	$X \sim \mathrm{B}(75, 0.075)$ $\mathrm{E}(X)=75\times 0.075=5.625,$ $\mathrm{Var}(X)=75(0.075)(1-0.075)=5.203125$ Since $n=30$ is large, by Central Limit Theorem, $\bar{X} \sim \mathrm{N}\left(5.625, \frac{5.203125}{30}\right)$ approximately Required probability $\mathrm{P}(\bar{X}\geq 5)$ $=0.933$ (3 sf)
10 (a)	
10 (b)	The product moment correlation coefficient for case (i) is 0.9965 (to 4 d.p.) while that of the case (ii) is -0.9182 (to 4 d.p.).

10 (c)	Considering the values of $\lvert r \rvert$, since the value for case (i) is closer to 1 as compared to that of case (ii), there is a stronger linear correlation between ln x and t as compared to e^{-x} and t. Hence case (i) is more appropriate. A suitable regression line of t on ln x will be $t = 1.309671 + 4.517503 \ln x$ $t = 1.31 + 4.52 \ln x$ (to 3 s.f.)
10 (d)	As x is the controlled variable, regression line of t on ln x should be used even though we are asked to estimate x. $t = 1.309671 + 4.517503 \ln x$ When $t = 8.5$, $x \approx 4.911896 \approx 4.91$ Since $r = 0.9965$ is close to +1 and that $t = 8.5$ is within the data range of t, the estimate of the value of quantity of chemical applied to the metal is reliable.
11 (a)	Unbiased estimate of the population mean $\bar{x} = \frac{186.35}{50} + 70$ $= 73.727$ Unbiased estimate of the population variance $s^2 = \frac{1}{49}\left(10494 - \frac{186.35^2}{50}\right)$ ≈ 199.989 ≈ 200 (3 s.f.)
11 (b)	$H_0 : \mu = 70$ $H_1 : \mu \neq 70$ Test at 5% significance level. Under H_0, since $n = 50$ is large, by Central Limit Theoremo, $\bar{X} \sim N\left(70, \frac{199.989}{50}\right)$ approximately, where $s^2 \approx 199.989$ is a good estimate of σ^2. Using GC, p-value = 0.0624 > 0.05 Do not reject H_0 and conclude that there is insufficient evidence, at 5% significance level, that the population mean mass of duck eggs is not equal to 70 grams.

	Hence Farm A's claim is valid.
11 (c)	No, it is not necessary to assume that X is normally distributed as the sample size of 50 is large. Central Limit Theorem could be applied and thus $\bar{X}$ is approximately normally distributed and a Z test can be carried out.
11 (d)	There is a probability of 0.05 that **the test will indicate** that the **population mean mass** of duck eggs is not 70 grams when in fact it is 70 grams.
	Unbiased estimate of the population variance $s^2 = \frac{40}{39}(146)$ $= 149.744$ Let Y be the mass, in grams, of a randomly chosen duck egg from Farm B . $H_0 : \mu = 70$ $H_1 : \mu > 70$ Test at 3% significance level. Under H_0 , since $n = 40$ is large, by Central Limit Theorem, $\bar{Y} \sim \mathrm{N}\left(70 , \frac{149.744}{40}\right)$ approximately, where $s^2 \approx 149.744$ is a good estimate of σ^2. OR or $\bar{Y} \sim \mathrm{N}\left(70 , \frac{146}{39}\right)$ approximately, where $s^2 = 40 \times \frac{146}{39}$ is a good estimate of σ^2. Since Farm B's claim is valid, we reject H_0. $\frac{k-70}{\sqrt{\frac{149.744}{40}}} \geq 1.8808$ OR $\frac{k-70}{\sqrt{\frac{146}{39}}} \geq 1.8808$ $k - 70 \geq 3.6390$ $k \geq 73.639$ $k \geq 73.6$ (3 s.f.) $\{k \in \mathbb{R} : k \geq 73.6\}$
12 (a)	Let W be the random variable denoting the mass of a randomly chosen packet of white sugar.

	$W \sim \mathrm{N}\left(1.05,\ 0.03^2\right)$ Let k kg be the mass exceeded by 80% of the packets of white sugar. $\mathrm{P}\left(W > k\right) = 0.8$ $k = 1.025 \quad (3 \text{ d.p.})$
12 (b)	98% of the labelled mass is 0.98 kg $\mathrm{P}\left(W < 0.98\right) = 0.0098153$ Let X be the number of packets of white sugar, out of 10, that are underweight. $X \sim \mathrm{B}\left(10,\ 0.0098153\right)$ $\mathrm{P}\left(X \le 1\right) = 0.996 \quad (3 \text{ s.f.})$
12 (c)	Let B be the random variable denoting the mass of a randomly chosen packet of brown sugar. $B \sim \mathrm{N}\left(m,\ 0.05^2\right)$ $\mathrm{P}\left(W_1 + W_2 + W_3 + W_4 > 2B\right) = 0.15$ Let $S = W_1 + W_2 + W_3 + W_4 - 2B$ $\mathrm{E}\left(S\right) = 4 \times 1.05 - 2m$ $= 4.2 - 2m$ $\mathrm{Var}\left(S\right) = 4 \times 0.03^2 + 2^2 \times 0.05^2$ $= 0.0136$ $S \sim N\left(4.2 - 2m\ ,\ 0.0136\right)$ $\mathrm{P}\left(S > 0\right) = 0.15$ $\mathrm{P}\left(Z > \dfrac{0 - 4.2 + 2m}{\sqrt{0.0136}}\right) = 0.15$ $\dfrac{2m - 4.2}{\sqrt{0.0136}} = 1.03643338$ $m = \dfrac{1}{2}\left(4.2 + 1.03643338\sqrt{0.0136}\right)$ $m \approx 2.16 \quad (3 \text{ s.f.})$
12 (d)	$B \sim \mathrm{N}\left(2.03,\ 0.05^2\right)$ Let $A = \dfrac{1}{9}\left(W_1 + W_2 + W_3 + W_4 + B_1 + B_2 + B_3 + B_4 + B_5\right)$ $\mathrm{E}\left(A\right) = \dfrac{1}{9}\left[4 \times 1.05 + 5 \times 2.03\right] = 1.59444$ $\mathrm{Var}\left(A\right) = \dfrac{1}{81}\left[4 \times 0.03^2 + 5 \times 0.05^2\right]$ $= \dfrac{1}{81}\left(0.0161\right) \quad or \quad 0.0001987654321$

	$A \sim \mathrm{N}\left(1.59444,\ \dfrac{0.0161}{81}\right)$ $\mathrm{P}(A > 1.6) = 0.347$ (3 s.f.)