1 On the same axes, sketch the graphs of $y=|x-a|$ and $y=|x-b|$, where a and b are constants such that $0<a<b$. You should show clearly the axial intercepts of both graphs. Hence solve the inequality $|x-a|<|x-b|$. [5]

2 **(a)** Without using a calculator, solve the inequality $\dfrac{17-5x}{x^2+5x-14}\geq -1$. [3]

(b) Hence solve the inequality $\dfrac{17x-5}{\frac{1}{x}+5-14x}\geq -1$. [3]

3 The non-zero vectors **a**, **b** and **c,** where **a**, **b** are non-parallel vectors, satisfy the equation $\mathbf{c}\times 3\mathbf{b}=5\mathbf{a}\times\mathbf{c}$.

(a) Determine, with clear reasons, the relationship between the vectors **c** and $5\mathbf{a}+3\mathbf{b}$. [4]

Referred to the origin *O*, it is given further that the vectors **a**, **b** and **c** are the position vectors of the points *A*, *B* and *C* respectively. Point *D* lies on the line segment *AB* such that it divides *AB* in the ratio $\lambda:1-\lambda$, where $0<\lambda<1$.

(b) Write down an expression for **d**, the position vector of point *D* in terms of **a** and **b**. [1]

The point *D* also lies on the line segment *OC*.

(c) Determine the exact value of λ. [3]

4 **(a)** The function f is defined by $\mathrm{f}:x\mapsto(x+3)^2-1$ for $x\in\mathbb{R},\ x\geq -3$.

Find $\mathrm{f}^{-1}(x)$, stating the domain of f^{-1}. [3]

(b) The function g is defined by

$$\mathrm{g}(x)=\begin{cases} x-1 & \text{for } 0<x<1,\\ x^2-1 & \text{for } 1\leq x\leq 2.\end{cases}$$

and $\mathrm{g}(x)=\mathrm{g}(x+2)$ for all values of *x*.

Sketch the graph of g for $-1\leq x<3$. Hence state the range of g. [3]

(c) The domain of g is now restricted to $-1\leq x<1$.

(i) Explain why the composite function $\mathrm{f}^{-1}\mathrm{g}$ exists. [1]

(ii) Find $\mathrm{f}^{-1}\mathrm{g}(x)$ in the form

$$\mathrm{f}^{-1}\mathrm{g}(x)=\begin{cases} \mathrm{p}(x) & \text{for } -1<x\leq 0,\\ \mathrm{q}(x) & \text{for } 0<x<1.\end{cases}$$

where $\mathrm{p}(x)$ and $\mathrm{q}(x)$ are expressions in terms of *x* to be determined. [3]

5 The curve C has equation

$$y=\frac{2x-6}{x^2+2x-3}.$$

(a) State the equations of the asymptotes of C. [2]

(b) Without using a calculator, find the range of values that y can take. [4]

(c) Sketch the graph of C, stating the equations of any asymptotes, the coordinates of the points where the curve crosses the axes and the stationary point(s). [4]

(d) Describe one transformation that will transform the curve C onto the curve $y=\frac{2x-8}{x^2-4}$. [1]

6 A curve C has parametric equations

$$x=\sqrt{23}\cos\left(t+\frac{\pi}{6}\right),\qquad y=2\sin t,\quad \text{where } 0\le t\le\pi.$$

(a) Find the exact Cartesian equation of l, the normal to C at the point $P\left(0,\sqrt{3}\right)$. [4]

(b) Let Q be a point on C such that the x-coordinate of Q is the minimum x-coordinate of C. The tangent to C at Q intersects l at point R. Find the exact coordinates of R. [3]

(c) Without using a graphing calculator, show that C has only one stationary point, and determine the nature of this stationary point. [4]

7 The line l_1 has equation $\mathbf{r}=11\mathbf{i}-3\mathbf{j}-\mathbf{k}+\lambda(\mathbf{i}-\mathbf{j}+2\mathbf{k})$, where λ is a parameter. The point A has coordinates $(2,0,-1)$.

(a) The plane p contains the line l_1 and the point A. Find a cartesian equation of the plane p. [3]

(b) Find the position vector of the point A', the reflection of the point A in the line l_1. [4]

The line l_2 passes through the point A, and is perpendicular to the line l_1. Planes p and q are perpendicular planes such that p and q meet at line l_2.

(c) Find a vector equation of the line l_2. [2]

(d) The plane Π is such that Π contains the point $(11,-3,-1)$ and is parallel to q. Find the perpendicular distance between Π and q. [2]

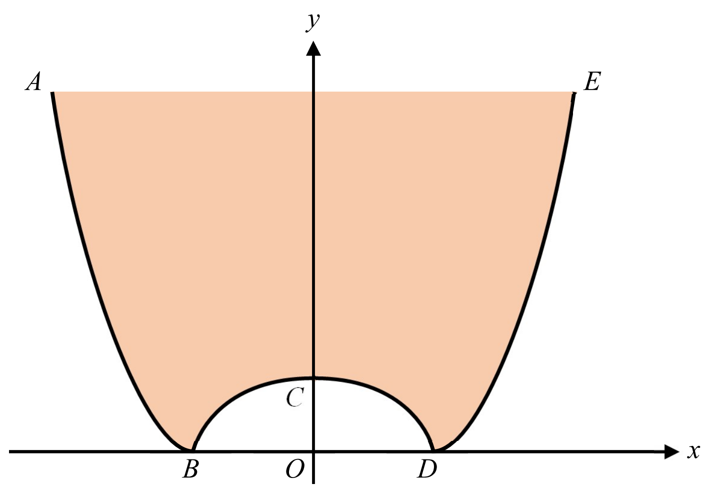
8 **(a)** A sequence $u_1, u_2, u_3, ...$ is such $u_{n+1} = 3u_n + An + B$, where A, B are constants and $n \geq 1$. Given that $u_1 = 2,\ u_2 = 5.5$ and $u_3 = 17.5$, find A and B, and find u_4. [4]

(b) It is given that $\mathrm{f}(r) = 2r^3 + 3r^2 + 4r + 5$.

(i) By considering $\mathrm{f}(r) - \mathrm{f}(r-1)$, show that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$. [5]

(ii) Given that $\sum_{r=1}^{n} r^3 = \frac{n^2(n+1)^2}{4}$, show that $\sum_{r=1}^{n} \mathrm{f}(r) = \frac{n(n+1)(n^2+3n+5)}{2} + Cn$, where C is a constant to be found. [3]

9 The diagram below, not drawn to scale, shows the vertical cross-section of a circular coffee cup with an indented base, passing through the axis of symmetry $x = 0$. The cup is filled with espresso to its brim. The curves AB and DE, at the body, form parts of the curve with equation $y = x^2 - 4$, while the indentation at the base BCD, forms part of the curve with equation $y = 2\cos\left(\frac{\pi x^2}{8}\right)$. The units of x and y are centimetres. The diameter of the rim is 7 cm and the cup has negligible thickness.

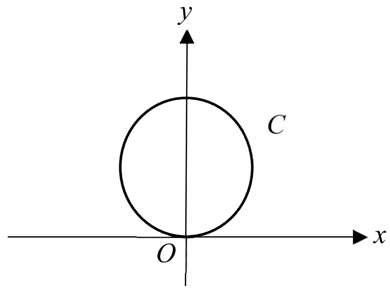


(a) Find the area of the cross-section of the cup shown in the diagram. [3]

(b) When the cup is empty, the indentation at the base can be viewed from the top. In order to conceal the indentation at the base, it is found that at least k cm^3 of espresso should be poured into the cup. Find the exact value of k. [6]

(c) When k cm^3 of espresso is already present in the cup, 14π cm^3 of hot water is further added to make a cup of Americano. Find the radius of the top surface of the Americano. [4]

10 A spherical container of radius 5 m is formed by rotating the following circle C about the y-axis.



The container has negligible thickness, and the circle C passes through the origin O.

(a) State a cartesian equation of C. [1]

Initially the spherical container is completely filled with water. Two engineers are calculating the time needed for the container to be completely drained from a small circular hole at the bottom. The volume of water in the container at time t seconds is denoted by V m^3.

[The volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.]

(b) The first engineer proposes that the rate of change of V with respect to t is a constant k.

(i) Write down a differential equation relating V, t and k. [1]

(ii) Determine, with justification, the sign of k. [1]

(iii) Find V in terms of t and k, leaving your answer in exact form. [2]

(c) The second engineer argues that the rate at which water flows out from the hole will be at its greatest in the beginning, and decreases as the depth of water in the container decreases. He suggests using Torricelli's law, which says that

$$\frac{dV}{dt} = -\alpha\sqrt{20h},$$

where α m^2 is the area of the circular hole at the bottom of the container, and h m is the depth of water in the container at time t seconds. The radius of the hole at the bottom of the container is found to be constant at 1 cm.

(i) Show that $\left(10h^{\frac{1}{2}} - h^{\frac{3}{2}}\right)\frac{dh}{dt} = -\frac{\sqrt{5}}{5000}$. [4]

(ii) Hence find the numerical value of t when the container is completely drained. [4]

Section A: Pure Mathematics [40 marks]

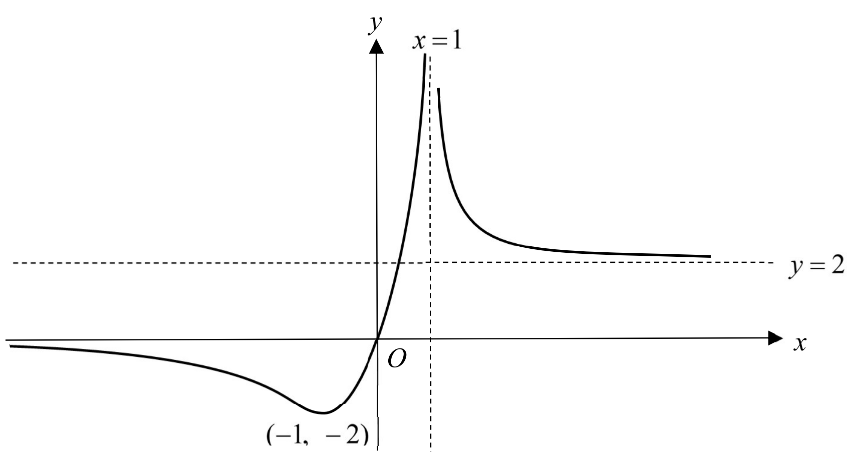
1 It is given that $f(x)=\sqrt{4-3x}$.

(a) Find the binomial expansion for $f(x)$, up to and including the term in x^2. Give the coefficients as exact fractions in their simplest form, and state the range of values of x for which this expansion is valid. [4]

(b) By taking $x=\frac{1}{4}$, show that $\sqrt{13}\approx\frac{1847}{512}$. [3]

(c) We may also substitute $x=\frac{1}{13}$ to obtain an approximation of $\sqrt{13}$. Without further calculation, explain why this leads to a better approximation of $\sqrt{13}$ than the value shown in **(b)**. [1]

2 The graph of $y=f(x)$ is given below. It has one vertical asymptote at $x=1$ and two horizontal asymptotes $y=0$ and $y=2$. The graph passes through the origin O and has a turning point at $(-1,\ -2)$.



(a) On separate diagrams, sketch the following graphs, indicating the coordinates of the points where the graphs cross the axes, the turning points, and the equations of any asymptotes if possible.

(i) $y=\frac{1}{f(x)}$ [3]

(ii) $y=f'(x)$ [3]

(iii) $y=f(|x-1|)$ [2]

(b) State the range of values of a such that $f(|x-1|)=a$ has only positive root(s). [1]

3 The sum S_n of the first n terms of a sequence u_1, u_2, u_3, … is given by

$$S_n = \ln\left(\frac{e^n}{3^{n^2}}\right).$$

(a) Show that $u_n = 1 + (1-2n)\ln 3$. [2]

(b) Hence, show that the sequence is an arithmetic progression. [2]

(c) Find the sum of the first ten odd-numbered terms. [2]

(d) A geometric sequence e^{u_1}, e^{u_2}, e^{u_3}, ... has a common ratio, r. Find the value of r. [2]

(e) Find the least value of n for which the sum of the first n terms of this geometric sequence is within 10^{-8} of its sum to infinity. [3]

4 **(a)** One of the roots of the equation $z^4 + pz^3 + 5z^2 + qz - 26 = 0$, where p and q are real, is $1 - 2\sqrt{3}\mathrm{i}$. Find the other roots of the equation, and the values of p and q. [5]

Do not use a calculator in answering part (b).

(b) The complex number w is given by $w = \dfrac{\mathrm{i}^3}{\left(-\sqrt{3}+\mathrm{i}\right)^4}$.

(i) Find the exact value of the modulus and argument of w. [4]

(ii) Find the smallest positive integer n such that $\dfrac{\mathrm{i}w^n}{w^*}$ is purely imaginary. [3]

Section B: Probability and Statistics [60 marks]

5 In this question you should state the parameters of any normal distribution you use.

The masses in grams of oranges have the distribution $\mathrm{N}(150, 14^2)$ and the masses in grams of kiwis have the distribution $\mathrm{N}(70, 8^2)$.

(a) Find the probability that the mass of a randomly chosen orange is less than 180 grams. [1]

6 oranges and 4 kiwis are randomly selected and packed into a randomly chosen empty basket to make a fruit basket. The masses of the empty baskets in grams have the distribution $\mathrm{N}(750, 168)$.

(b) Find the probability that a randomly chosen fruit basket is within 25 grams of its mean. [3]

(c) Sketch the distribution for masses of fruit baskets between 1710 grams and 2150 grams. [2]

(d) Three fruit baskets were randomly chosen. Find the probability that exactly one fruit basket weighs more than 2000 grams and exactly one fruit basket weighs less than 1900 grams. [2]

6 A company manufactures screen protectors for handphones. On average, 17% of the screen protectors are cracked. The screen protectors are packaged into boxes of 20. It should be assumed that the number of cracked screen protectors in a box of 20 screen protectors follows a binomial distribution.

(a) Show that the probability of having no more than 3 cracked screen protectors in a randomly chosen box is 0.55041, correct to 5 significant figures. [1]

(b) Find the most probable number of cracked screen protectors in a randomly chosen box. [1]

Every month, the company exports a large shipment of *n* cartons of screen protectors. In each carton, there are 8 boxes of screen protectors. Every carton is assumed to be filled completely.

(c) Using a suitable approximation, find the least value of *n* such that in a randomly chosen month, there is at least a 99% chance that the average number of boxes per carton having no more than 3 cracked screen protectors is at most 5. [4]

To determine whether cartons are ready for local or overseas sales, cartons are randomly selected for inspection. In a carton,

- if there are at least 4 boxes that contain no more than 3 cracked screen protectors each, the carton is ready for local sales;
- if there are at least 6 boxes that contain no more than 3 cracked screen protectors each, the carton is ready for overseas sales.

(d) A carton that is not ready for overseas sales is randomly selected. Find the probability that the carton is ready for local sales. [3]

7 In a carnival lucky dip game, a game master places *n* consolation tickets, *m* blank tickets, and one golden ticket into a box. A contestant taking part in the game would pay \$1 to draw two tickets from the box. They would then be awarded \$1 for each consolation ticket drawn, and \$10 for the golden ticket if it is drawn. Nothing is awarded for the blank tickets drawn. Let \$*W* represent the total amount awarded to a contestant after one game.

(a) Show that $\mathrm{P}(W=1)=\dfrac{2nm}{(n+m+1)(n+m)}$, and determine the probability distribution of *W*. [4]

(b) By considering $\mathrm{E}(W)$, show that if the game master expects to make a profit from the lucky dip game, then $m-n>19$. [3]

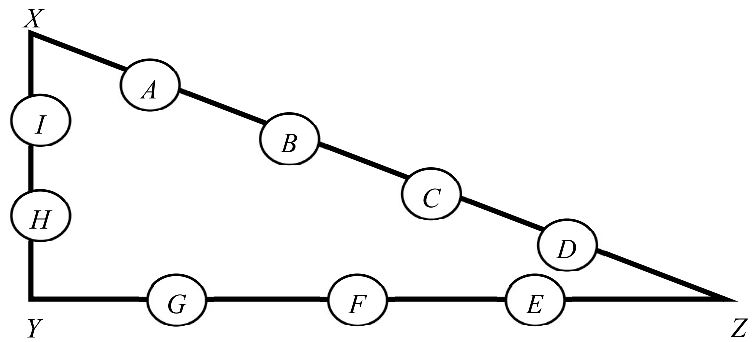
The game master then decides to run the game with 10 consolation tickets and 40 blank tickets. He is also issued with carnival lucky draw tickets to give away, and gives each contestant *Y* lucky draw tickets after the game, where $Y=|W-4|$.

(c) Find $\mathrm{E}(Y)$ and $\mathrm{Var}(Y)$. [2]

8 In an art installation, three points on the floor, X, Y and Z, were connected using a rope to form a triangle XYZ. 9 bulbs, labelled A, B, C, D, E, F, G, H, I, are then to be placed along the rope to improve the aesthetic appearance.

(a) The 9 bulbs are first distributed into 3 groups, namely Group XY, Group YZ and Group ZX. Find the number of ways the bulbs can be distributed if each group consists of at least 2 bulbs. [2]

In one particular distribution from part **(a)**, the bulbs from Group XY, Group YZ and Group ZX are placed on the sides XY, YZ and ZX of the triangle XYZ respectively. The bulbs are arranged as shown in the diagram below.



(b) By connecting the bulbs in this arrangement using additional rope, we can form a smaller triangle within the triangle XYZ. For example, the bulbs B, I, G form a smaller triangle when connected. How many different smaller triangles can be formed? [2]

(c) Each bulb can be programmed to produce either red, blue or green light when it is switched on. Find the number of ways the colours can be programmed so that all the 3 colours are used. [3]

In another art installation, 9 bulbs, labelled A, A, B, B, C, C, D, E, F, are laid on the floor in a circular arrangement.

(d) Find the number of possible arrangements if the 2 A's are separated and the 2 B's are separated. [3]

9 Orange is a company that produces laptops and tablets. The company has a patented design for their ultra-slim laptop cooling fans, which operate at an optimal speed of 3100 RPM, such that they maximise the cooling effectiveness while minimising the noise level.

In a routine quality check by the company's internal surveyors, a random sample of 13 laptops were tested and their cooling fan speeds are recorded in the table below.

3202	3033	3103	3013	3023	3032	3145	3054	3154	3083	3099	3101	3102

(a) Calculate the unbiased estimates of the population mean and population variance of the fan speeds. [2]

Based on historical data, it is known that the fan speeds have a standard deviation of 30 RPM.

(b) Stating any necessary assumptions, test at the 10% level of significance whether the population mean fan speed is different from the optimal speed of 3100 RPM. [6]

(c) State the meaning of the p-value in this context. [1]

(d) State, to the nearest percent, the smallest level of significance at which the null hypothesis will be rejected using the given set of data. [1]

A random sample of 50 tablets is tested and the standby-time, in hours, is measured. It is known that the standby-time of the tablets is normally distributed with a standard deviation of 3 hours. Furthermore, Orange markets their tablets as having a standby-time of 36 hours.

(e) Determine the range of values for the sample mean standby-time, in hours, such that the null hypothesis will be rejected in a test at the 5% level of significance against an alternative hypothesis that the mean standby-time is less than 36 hours. [2]

10 A group of researchers want to study the relationship between the birth lengths of newborns and the heights of these newborns when they are 16 years old. They record the birth lengths of 10 newborns, l centimetres, and their respective heights, h centimetres, when they are 16 years old.

Birth length, l cm	40.2	41.1	41.8	42.4	45.6	46.0	46.5	48.5	50.2	51.8
Height at 16 years old, h cm	160.5	160.8	161.1	161.9	165.2	176.7	166.5	171.9	178.4	187.6

It turns out that one of the values of h was incorrectly recorded.

(a) Sketch a scatter diagram for the data and circle the point that was incorrectly recorded. [2]

For parts **(b)**, **(c),** and **(d)** of this question, you should **exclude** this incorrectly recorded point.

(b) Use your scatter diagram to explain whether the relationship between l and h is likely to be well-modelled by an equation of the form $h = a + bl$, where a and b are constants. [1]

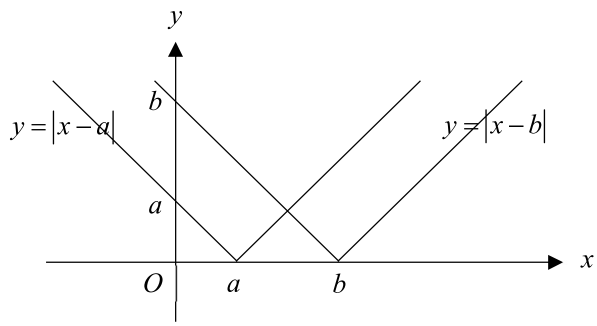
(c) By calculating the relevant product moment correlation coefficients, determine whether the relationship between l and h is modelled better by $h = a + bl$ or $h = a + bl^3$. Explain how you decide which model is better, and state the equation of the regression line in this case. [5]

(d) Explain whether or not the regression line in part **(c)** should be used to estimate the birth length of a 16-year-old whose height is 170 cm. [1]

(e) Use the regression line in part **(c)** to estimate the value of h for the incorrectly recorded point identified in part **(a)** to 1 decimal place. Comment on the reliability of this estimate. [2]

(f) Give a reason in context why, although there is a strong positive correlation between one's birth length and one's height at 16 years old, we cannot say that one's birth length is the cause of one's height at 16 years old. [1]

1



Point of intersection:

$x-a=-(x-b)$

$x=\dfrac{a+b}{2}$

$\therefore x<\dfrac{a+b}{2}$

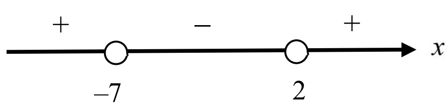
2(a) $\dfrac{17-5x}{x^2+5x-14}+1\geq 0$

$\dfrac{x^2+3}{x^2+5x-14}\geq 0$

$\dfrac{x^2+3}{(x+7)(x-2)}\geq 0$

As $x^2+3>0$ for all real values of x,

$(x+7)(x-2)>0$



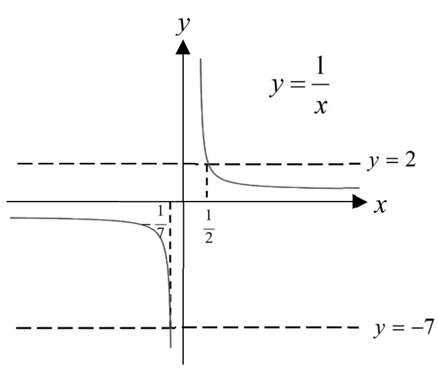
$x<-7$ OR $x>2$

2(b) Dividing numerator and denominator of fraction by x:

$\dfrac{17-\frac{5}{x}}{\frac{1}{x^2}+\frac{5}{x}-14}\geq -1$

Replacing x in (a) with $\dfrac{1}{x}$:

$\dfrac{1}{x}<-7$ OR $\dfrac{1}{x}>2$



$-\dfrac{1}{7}<x<0$ OR $0<x<\dfrac{1}{2}$

3(a) $\mathbf{c}\times 3\mathbf{b} = 5\mathbf{a}\times\mathbf{c}$

$\Rightarrow \mathbf{0} = (5\mathbf{a}\times\mathbf{c}) - (\mathbf{c}\times 3\mathbf{b}) = (5\mathbf{a}\times\mathbf{c}) + (3\mathbf{b}\times\mathbf{c})$

$\Rightarrow (5\mathbf{a}+3\mathbf{b})\times\mathbf{c} = \mathbf{0}$

Either $\mathbf{c}=\mathbf{0}$ or $5\mathbf{a}+3\mathbf{b}=\mathbf{0}$ or $\mathbf{c}$ is parallel to $5\mathbf{a}+3\mathbf{b}$

$\mathbf{c}\neq\mathbf{0}$, and since **a** and **b** are non-parallel, non-zero vectors, $5\mathbf{a}+3\mathbf{b}\neq\mathbf{0}$

Hence **c** is parallel to $5\mathbf{a}+3\mathbf{b}$.

(b) $\mathbf{d} = \dfrac{(1-\lambda)\mathbf{a}+\lambda\mathbf{b}}{\lambda+(1-\lambda)} = (1-\lambda)\mathbf{a}+\lambda\mathbf{b}$

(c) <u>Method 1 (Comparing Coefficients)</u>

Since D lies on OC, **d** is parallel to both **c** and $5\mathbf{a}+3\mathbf{b}$

Hence $\mathbf{d} = (1-\lambda)\mathbf{a}+\lambda\mathbf{b} = k(5\mathbf{a}+3\mathbf{b})$

Since **a** and **b** are non-parallel vectors, by comparing coefficients,

$1-\lambda = 5k$ and $\lambda = 3k$

Solving, $\lambda = \dfrac{3}{8}$ $\left(\text{and} \quad k = \dfrac{1}{8}\right)$

<u>Method 2 (Cross Product)</u>

Since D lies on OC, **d** is parallel to **c** and $5\mathbf{a}+3\mathbf{b}$

$\therefore \mathbf{d}\times(5\mathbf{a}+3\mathbf{b}) = \mathbf{0}$

$[(1-\lambda)\mathbf{a}+\lambda\mathbf{b}]\times(5\mathbf{a}+3\mathbf{b}) = \mathbf{0}$

$5(1-\lambda)\mathbf{a}\times\mathbf{a} + 5\lambda\mathbf{b}\times\mathbf{a} + 3(1-\lambda)\mathbf{a}\times\mathbf{b} + 3\lambda\mathbf{b}\times\mathbf{b} = \mathbf{0}$

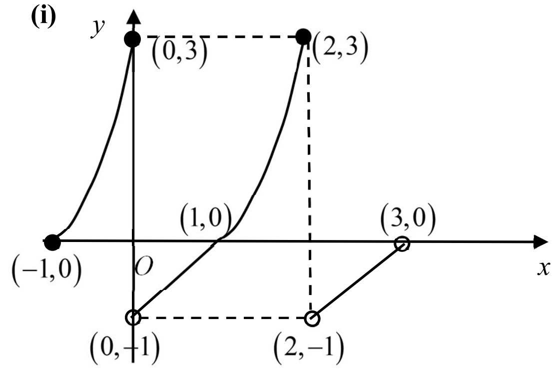
As $\mathbf{a}\times\mathbf{a}, \mathbf{b}\times\mathbf{b} = \mathbf{0}$, and $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$:

$-5\lambda\mathbf{a}\times\mathbf{b} + 3(1-\lambda)\mathbf{a}\times\mathbf{b} = \mathbf{0}$

$(3-8\lambda)\mathbf{a}\times\mathbf{b} = \mathbf{0}$

a and **b** are non-parallel $\Rightarrow \mathbf{a}\times\mathbf{b}\neq\mathbf{0}$

$\therefore 3-8\lambda = 0, \quad \lambda = \dfrac{3}{8}$

4(a)	Let $y=(x+3)^2-1$ $y+1=(x+3)^2$ $x=-3+\sqrt{y+1}$ or $-3-\sqrt{y+1}$ (rej $\because x\geq -3$) So $\mathrm{f}^{-1}(x)=-3+\sqrt{x+1}$ $D_{\mathrm{f}^{-1}}=R_{\mathrm{f}}=[-1,\infty)$.
(b)	**(i)**  $\mathrm{R_g}=(-1,3]$
(c)(i)	Note that $R_{\mathrm{g}}=(-1,3]$ and $D_{\mathrm{f}^{-1}}=[-1,\infty)$. Since $R_{\mathrm{g}}\subseteq D_{\mathrm{f}^{-1}}$, $\mathrm{f}^{-1}\mathrm{g}$ exists.
(c)(ii)	For $0<x<1$, $\mathrm{f}^{-1}\mathrm{g}(x)=-3+\sqrt{(x-1)+1}=-3+\sqrt{x}$ For $-1\leq x<0$, note that $\mathrm{g}(x)=(x+2)^2-1$. Hence, $\mathrm{f}^{-1}\mathrm{g}(x)=-3+\sqrt{(x+2)^2-1+1}=-3+x+2=x-1$ $\mathrm{f}^{-1}\mathrm{g}(x)=\begin{cases}x-1 & \text{for } -1<x\leq 0\\ -3+\sqrt{x} & \text{for } 0<x<1\end{cases}$ Hence, $\mathrm{p}(x)=x-1$ and $\mathrm{q}(x)=-3+\sqrt{x}$

5(a) Horizontal Asymptote:

$y=0$

Vertical Asymptote:

$x^2+2x-3=(x+3)(x-1)$

Vertical asymptotes are $x=-3,\ x=1$.

(b) $yx^2+2yx-3y=2x-6$

$yx^2+(2y-2)x+(6-3y)=0$

Since $x\in\mathbb{R}$,

Discriminant $(2y-2)^2-4y(6-3y)\geq 0$

$16y^2-32y+4\geq 0$

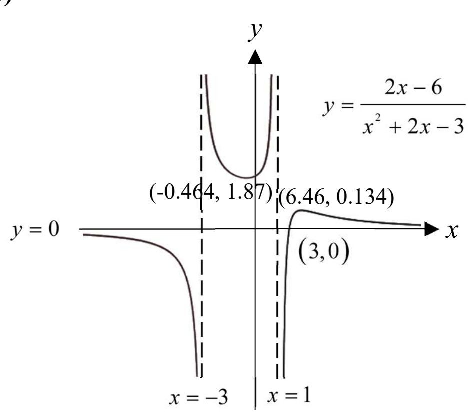
$4y^2-8y+1\geq 0$

As roots of $4y^2-8y+1=0$ are:

$$y=\frac{8\pm\sqrt{8^2-4(4)(1)}}{2(4)}=1\pm\frac{\sqrt{3}}{2}\ ,$$

$$y\leq 1-\frac{\sqrt{3}}{2}\ \text{ OR }\ y\geq 1+\frac{\sqrt{3}}{2}\ .$$

(c)



(d) Translation of *C* by 1 unit in the positive *x*-direction.

6(a)	$x=\sqrt{23}\cos\left(t+\frac{\pi}{6}\right)$, $y=2\sin t$ At $\left(0,\sqrt{3}\right)$, $t=\frac{\pi}{3}$ $\frac{\mathrm{d}x}{\mathrm{d}t}=-\sqrt{23}\sin\left(t+\frac{\pi}{6}\right)$, $\frac{\mathrm{d}y}{\mathrm{d}t}=2\cos t$ $\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{2\cos t}{-\sqrt{23}\sin\left(t+\frac{\pi}{6}\right)}$ When $t=\frac{\pi}{3}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{1}{\sqrt{23}}$ Hence, gradient of normal $=\sqrt{23}$ Equation of normal: $y=\sqrt{23}x+\sqrt{3}$
(b)	$x=\sqrt{23}\cos\left(t+\frac{\pi}{6}\right)$ To find min x-coordinate, <u>Method 1 (range of values of cosine)</u> Note that $-1\le\cos\left(t+\frac{\pi}{6}\right)\le 1$. Hence, minimum x-coordinate occurs when $x=-\sqrt{23}$. This occurs when $t=\frac{5\pi}{6}$. <u>Method 2 (differentiation)</u> $\frac{\mathrm{d}x}{\mathrm{d}t}=-\sqrt{23}\sin\left(t+\frac{\pi}{6}\right)=0$ $\Rightarrow t=\frac{5\pi}{6}$ or $-\frac{\pi}{6}$(rejected $\because 0\le t\le\pi$) When $t=\frac{5\pi}{6}$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ is undefined. Hence the tangent of curve at $t=\frac{5\pi}{6}$ is **parallel to the *y*-axis**. Therefore, equation of tangent is $x=-\sqrt{23}$ When tangent intersects l, $x=-\sqrt{23}\Rightarrow y=\sqrt{23}\left(-\sqrt{23}\right)+\sqrt{3}=\sqrt{3}-23$ $\therefore R\left(-\sqrt{23},\sqrt{3}-23\right)$

(c) $\dfrac{dy}{dx} = \dfrac{2\cos t}{-\sqrt{23}\sin\left(t+\dfrac{\pi}{6}\right)}$

For stationary points, $\dfrac{dy}{dx} = 0$:

$\dfrac{2\cos t}{-\sqrt{23}\sin\left(t+\dfrac{\pi}{6}\right)} = 0 \Rightarrow \cos t = 0$

Since $0 \le t \le \pi$, then $t = \dfrac{\pi}{2}$

Since there is **only one** value of t such that $\dfrac{dy}{dx} = 0$, then C has only one stationary point. (Shown)

When $t = \dfrac{\pi}{2}$, $x = \sqrt{23}\cos\left(\dfrac{\pi}{2}+\dfrac{\pi}{6}\right) = \dfrac{-\sqrt{23}}{2}$

x	$\left(\dfrac{-\sqrt{23}}{2}\right)^-$	$\dfrac{-\sqrt{23}}{2}$	$\left(\dfrac{-\sqrt{23}}{2}\right)^+$
t	$\left(\dfrac{\pi}{2}\right)^+$ (for e.g. 1.58)	$\dfrac{\pi}{2}$	$\left(\dfrac{\pi}{2}\right)^-$ (for e.g. 1.56)
Value of $\dfrac{dy}{dx}$	0.00445577	0	-0.0051669
Explain	$\cos t < 0$ and $\sin\left(t+\dfrac{\pi}{6}\right) > 0$	0	$\cos t > 0$ and $\sin\left(t+\dfrac{\pi}{6}\right) > 0$
Sign of $\dfrac{dy}{dx}$	+	0	-
Nature of stationary value	Maximum value		

Hence, maximum value occurs when $t = \dfrac{\pi}{2}$.

7(a) Two vectors parallel to p are $\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $\begin{pmatrix}2\\0\\-1\end{pmatrix}-\begin{pmatrix}11\\-3\\-1\end{pmatrix}=3\begin{pmatrix}-3\\1\\0\end{pmatrix}$.

$$\begin{pmatrix}1\\-1\\2\end{pmatrix}\times\begin{pmatrix}-3\\1\\0\end{pmatrix}=\begin{pmatrix}-2\\-6\\-2\end{pmatrix}=-2\begin{pmatrix}1\\3\\1\end{pmatrix}.$$

So a vector normal to p is $\begin{pmatrix}1\\3\\1\end{pmatrix}$.

Since $A(2,0,-1)$ lies on p, equation of p is $\mathbf{r}\cdot\begin{pmatrix}1\\3\\1\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\3\\1\end{pmatrix}=1$.

So a cartesian equation of p is $x+3y+z=1$.

(b) Let G be the foot of perpendicular from A to l_1.

Since G lies on l_1, $\overrightarrow{OG}=\begin{pmatrix}11\\-3\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ for some λ.

$$\overrightarrow{AG}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0\Rightarrow\left[\begin{pmatrix}11\\-3\\-1\end{pmatrix}+\lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}-\begin{pmatrix}2\\0\\-1\end{pmatrix}\right]\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0$$

$$\Rightarrow 12+6\lambda-0=0$$

$$\Rightarrow \lambda=-2$$

Hence, $\overrightarrow{OG}=\begin{pmatrix}9\\-1\\-5\end{pmatrix}$.

Using Ratio Theorem, $\overrightarrow{OG}=\dfrac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$.

$$\therefore\quad \overrightarrow{OA'}=2\overrightarrow{OG}-\overrightarrow{OA}=2\begin{pmatrix}9\\-1\\-5\end{pmatrix}-\begin{pmatrix}2\\0\\-1\end{pmatrix}=\begin{pmatrix}16\\-2\\-9\end{pmatrix}.$$

<u>Alternative solution (using projection vector to find $\overrightarrow{OG}$)</u>

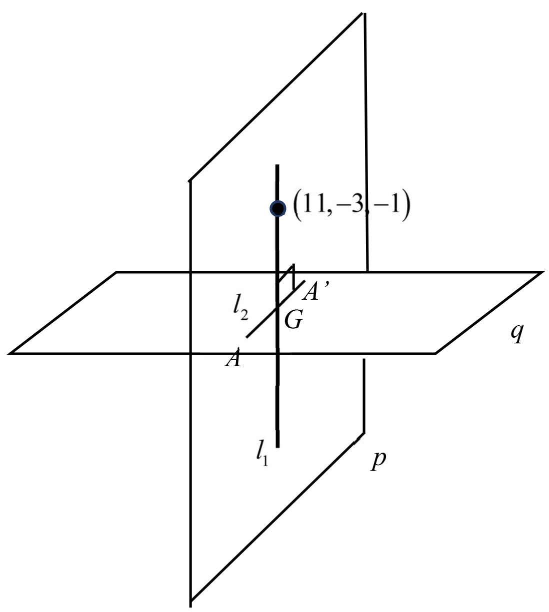
$$\overrightarrow{BG}=\left[\overrightarrow{BA}\cdot\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\2\end{pmatrix}\right]\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$=\frac{1}{6}\left[\begin{pmatrix}-9\\3\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}\right]\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$=-2\begin{pmatrix}1\\-1\\2\end{pmatrix}$$

$$\therefore\ \overrightarrow{OG}=-2\begin{pmatrix}1\\-1\\2\end{pmatrix}+\begin{pmatrix}11\\-3\\-1\end{pmatrix}=\begin{pmatrix}9\\-1\\-5\end{pmatrix}.$$

(c)



A direction vector of l_2 is $\overrightarrow{AG}=\begin{pmatrix}7\\-1\\-4\end{pmatrix}$.

A vector equation of l_2 is $\mathbf{r}=\begin{pmatrix}2\\0\\-1\end{pmatrix}+\mu\begin{pmatrix}7\\-1\\-4\end{pmatrix},\quad \mu\in\mathbb{R}$.

<u>Alternative solution (consider l_2 as intersection of *p* and *q*)</u>

$x+3y+z=1$ --- (1)

$$\mathbf{r}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}2\\0\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0\Rightarrow x-y+2z=0 \quad \text{--- (2)}$$

Using GC to solve (1) and (2),

A equation of l_2 is $\mathbf{r}=\begin{pmatrix}\frac{1}{4}\\\frac{1}{4}\\0\end{pmatrix}+\mu\begin{pmatrix}-\frac{7}{4}\\\frac{1}{4}\\1\end{pmatrix},\quad \mu\in\mathbb{R}$.

(d) Note that the points $G(9,-1,-5)$ on q and $B(11,-3,-1)$ on $\prod$, lie on the same line l_1, and l_1 is perpendicular to both q and $\prod$.

Perpendicular distance between $\prod$ and q $=\left|\overrightarrow{GB}\right|=\left|\begin{pmatrix}11\\-3\\-1\end{pmatrix}-\begin{pmatrix}9\\-1\\-5\end{pmatrix}\right|=\left|\begin{pmatrix}2\\-2\\4\end{pmatrix}\right|=2\sqrt{6}$ units

<u>Alternative solution (using formula)</u>

Plane $\prod$: $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=\begin{pmatrix}11\\-3\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=12$

Plane q: $\mathbf{r}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}=0$

Perpendicular distance $=\dfrac{|d_1-d_2|}{|\mathbf{n}|}=\dfrac{|12-0|}{\left|\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|}=\dfrac{12}{\sqrt{6}}$ unit

<u>Alternative solution (using projection of $\overrightarrow{AB}$ on normal)</u>

Perp dist $=\dfrac{\left|\overrightarrow{AB}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|}=\dfrac{\left|\begin{pmatrix}9\\-3\\0\end{pmatrix}\cdot\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|}{\left|\begin{pmatrix}1\\-1\\2\end{pmatrix}\right|}=\dfrac{12}{\sqrt{6}}$

units

<u>Alternative solution (using cross product)</u>

Perpendicular distance $=\dfrac{\left|\overrightarrow{AB}\times\overrightarrow{AG}\right|}{\left|\overrightarrow{AG}\right|}$

$=\dfrac{\left|\begin{pmatrix}9\\-3\\0\end{pmatrix}\times\begin{pmatrix}7\\-1\\-4\end{pmatrix}\right|}{\left|\begin{pmatrix}7\\-1\\-4\end{pmatrix}\right|}=\dfrac{\left|\begin{pmatrix}12\\36\\12\end{pmatrix}\right|}{\left|\begin{pmatrix}7\\-1\\-4\end{pmatrix}\right|}=\dfrac{\sqrt{1584}}{\sqrt{66}}=\sqrt{24}$ units

8(a) When $n=1$ $u_2=3(2)+A(1)+B=5.5 \Rightarrow A+B=-0.5$ ---(1)

When $n=2$, $u_3=3(5.5)+A(2)+B=17.5 \Rightarrow 2A+B=1$ ---(2)

Solving, $A=1.5, B=-2$

So $u_{n+1}=3u_n+1.5n-2$

When $n=3$,

$$\begin{aligned} u_4 &= 3u_3+1.5(3)-2 \\ &= 3(17.5)+4.5-2 \\ &= 55 \end{aligned}$$

(b)(i) $\mathrm{f}(r)-\mathrm{f}(r-1)=\left(2r^3+3r^2+4r+5\right)$

$$-\left(2(r-1)^3+3(r-1)^2+4(r-1)+5\right)$$

$$\begin{aligned} &=\left(2r^3+3r^2+4r+5\right) \\ &\quad -\left(2r^3-6r^2+6r-2+3r^2-6r+3+4r-4+5\right) \\ &=6r^2+3 \end{aligned}$$

$\therefore \mathrm{f}(r)-\mathrm{f}(r-1)=6r^2+3$

$$\Rightarrow \sum_{r=1}^{n}\left(6r^2+3\right)=\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]$$

$$\Rightarrow 6\left(\sum_{r=1}^{n}r^2\right)+3n=\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]$$

$$\Rightarrow \sum_{r=1}^{n}r^2=\frac{1}{6}\left\{\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]-3n\right\}$$

$$\sum_{r=1}^{n}\left[\mathrm{f}(r)-\mathrm{f}(r-1)\right]=\left[\begin{array}{l} \mathrm{f}(1)-\mathrm{f}(0) \\ +\mathrm{f}(2)-\mathrm{f}(1) \\ +\mathrm{f}(3)-\mathrm{f}(2) \\ \vdots \\ +\mathrm{f}(n-2)-\mathrm{f}(n-3) \\ +\mathrm{f}(n-1)-\mathrm{f}(n-2) \\ +\mathrm{f}(n)-\mathrm{f}(n-1) \end{array}\right]$$

$$=\mathrm{f}(n)-\mathrm{f}(0)$$

Then $\sum_{r=1}^{n}r^2=\frac{1}{6}\left[\mathrm{f}(n)-\mathrm{f}(0)-3n\right]$

$$\Rightarrow \quad \sum_{r=1}^{n} r^2 = \frac{2n^3+3n^2+4n+5-5-3n}{6}$$

$$= \frac{n\left(2n^2+3n+1\right)}{6}$$

$$= \frac{n(n+1)(2n+1)}{6} \text{ (shown)}$$

(b)(ii) $\sum_{r=1}^{n} \mathrm{f}(r) = \sum_{r=1}^{n}\left(2r^3+3r^2+4r+5\right)$

$$= 2\sum_{r=1}^{n}\left(r^3\right) + 3\sum_{r=1}^{n}\left(r^2\right) + 4\sum_{r=1}^{n}(r) + \sum_{r=1}^{n}(5)$$

$$= 2\left[\frac{n^2(n+1)^2}{4}\right] + 3\left[\frac{n(n+1)(2n+1)}{6}\right] + 4\left[\frac{n(n+1)}{2}\right] + 5n$$

$$= \frac{\left[n(n+1)\right]\left[n(n+1)\right]}{2} + \frac{n(n+1)(2n+1)}{2} + \frac{4n(n+1)}{2} + 5n$$

$$= \frac{n(n+1)\left[n(n+1)+2n+1+4\right]}{2} + 5n$$

$$= \frac{n(n+1)\left(n^2+3n+5\right)}{2} + 5n \text{ (shown)}$$

9(a) Note that the respective coordinates are $C(0,2)$, $D(2,0)$ and $E(3.5,8.25)$.

Method 1 (area w.r.t. *y*-axis)

$$\text{Area of cross-section} = 2\left[\int_0^{8.25}\sqrt{y+4}\ \mathrm{d}y - \int_0^2\sqrt{\frac{8}{\pi}\cos^{-1}\frac{y}{2}}\ \mathrm{d}y\right]$$

$$= 40.3\ \text{cm}^2$$

Method 2 (mixed area between *x*- and *y*-axis)

$$\text{Area of cross-section} = 2\left[\int_0^{8.25}\sqrt{y+4}\ \mathrm{d}y - \int_0^2 2\cos\left(\frac{\pi x^2}{8}\right)\mathrm{d}x\right]$$

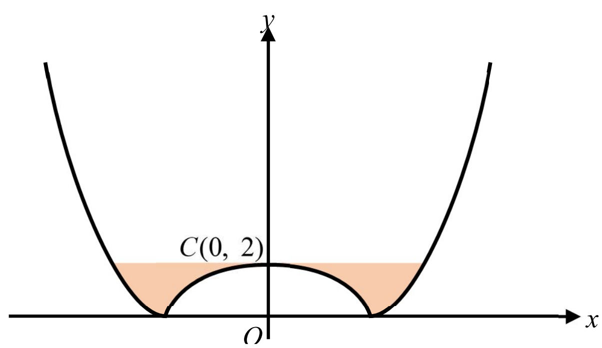
$$= 40.3\ \text{cm}^2$$

Method 3 (area w.r.t. x-axis)

$$\text{Area of cross-section} = 2\left[3.5(8.25) - \int_0^2 2\cos\left(\frac{\pi x^2}{8}\right)\mathrm{d}x - \int_2^{3.5}(x^2-4)\ \mathrm{d}x\right]$$

$$= 40.3\ \text{cm}^2$$

(b)



When volume is k cm^3, espresso level just touches $C(0,\ 2)$.

Method 1: Integration by parts

$$k = \pi\int_0^2 (y+4)\ \mathrm{d}y - \pi\int_0^2 \frac{8}{\pi}\cos^{-1}\frac{y}{2}\ \mathrm{d}y$$

$$= \pi\left[\frac{y^2}{2}+4y\right]_0^2 - 8\int_0^2\cos^{-1}\frac{y}{2}\ \mathrm{d}y$$

$$= 10\pi - 8\left\{\left[y\cos^{-1}\frac{y}{2}\right]_0^2 + \frac{1}{2}\int_0^2\frac{y}{\sqrt{1-(\frac{y}{2})^2}}\ \mathrm{d}y\right\}$$

$$= 10\pi + 16\left[\sqrt{1-(\tfrac{y}{2})^2}\right]_0^2$$

$$= 10\pi - 16$$

Method 2: Integration using substitution

$$k=\pi\int_0^2 (y+4)\ \mathrm{d}y-\pi\int_0^2 \frac{8}{\pi}\cos^{-1}\frac{y}{2}\ \mathrm{d}y$$

$$=\pi\left[\frac{y^2}{2}+4y\right]_0^2-8\int_0^2 \cos^{-1}\frac{y}{2}\ \mathrm{d}y$$

$$=10\pi-8\int_0^2 \cos^{-1}\frac{y}{2}\ \mathrm{d}y$$

Let $u=\cos^{-1}\frac{y}{2}\Rightarrow y=2\cos u$

$$\frac{\mathrm{d}y}{\mathrm{d}u}=-2\sin u$$

When $y=0,\ u=\frac{\pi}{2}$

$y=2,\ u=0$

$$\int_0^2 \cos^{-1}\frac{y}{2}\ \mathrm{d}y=\int_{\frac{\pi}{2}}^0 u\ \left(-2\sin u\ \mathrm{d}u\right)$$

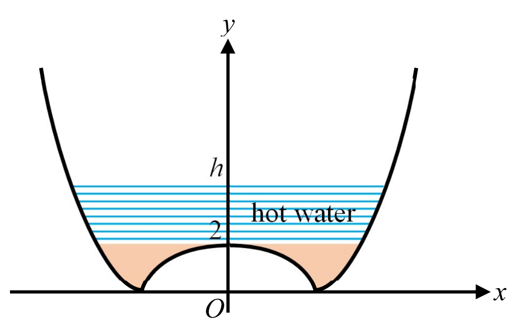
$$=\int_0^{\frac{\pi}{2}} 2u\sin u\ \mathrm{d}u$$

$$=\left[-2u\cos u\right]_0^{\frac{\pi}{2}}+\int_0^{\frac{\pi}{2}} 2\cos u\ \mathrm{d}u$$

$$=2$$

$$\therefore k=10\pi-8(2)=10\pi-16$$

(c)



Let the height be h cm after hot water is added.

Volume of hot water $=\pi\int_2^h (y+4)\ \mathrm{d}y$

$$=\pi\left[\frac{y^2}{2}+4y\right]_2^h=14\pi$$

$$\Rightarrow \frac{h^2}{2}+4h-10=14$$

$$\Rightarrow h^2+8h-48=0$$

Using GC, $h=4$ or -12 (rej.).

$\therefore$ Radius of the top surface $=\sqrt{h+4}=2\sqrt{2}$ cm.

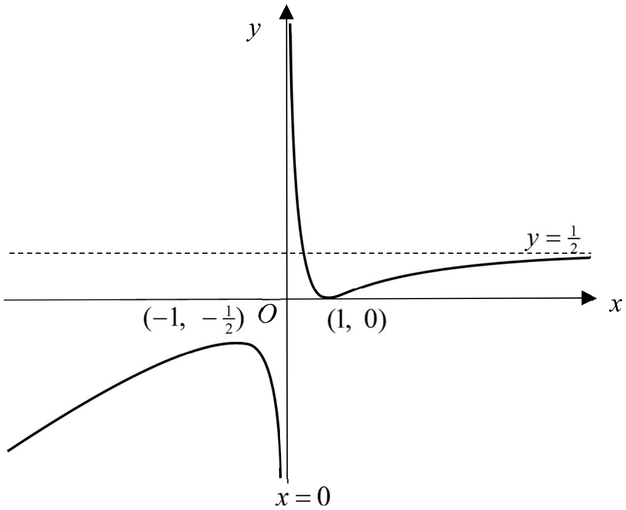
10(a)	$x^2+(y-5)^2=25$
(b)(i)	$\frac{\mathrm{d}V}{\mathrm{d}t}=k$
(ii)	As the volume of water in the container is decreasing with time, $k<0$.
(iii)	$\int \frac{\mathrm{d}V}{\mathrm{d}t}\,\mathrm{d}t=\int k\ \mathrm{d}t$ $\therefore V=kt+C$, where C is an arbitrary constant When $t=0$, $\frac{4}{3}\pi(5)^3=k(0)+C\Rightarrow C=\frac{500\pi}{3}$
(c)(i)	Area of hole, $\alpha=\pi\left(\frac{1}{100}\right)^2=\frac{\pi}{10000}\mathrm{m}^2$ So $\frac{\mathrm{d}V}{\mathrm{d}t}=-\frac{\pi}{10000}\sqrt{20h}=-\frac{\sqrt{5}\pi}{5000}\sqrt{h}$ By Chain Rule, $\frac{\mathrm{d}V}{\mathrm{d}h}\times\frac{\mathrm{d}h}{\mathrm{d}t}=\frac{\mathrm{d}V}{\mathrm{d}t}$ (so we need to find $\frac{\mathrm{d}V}{\mathrm{d}h}$) <u>Method 1 (use direct expression for *V*)</u> $V=\pi\int_0^h 25-(y-5)^2\ \mathrm{d}y$ $=\pi\int_0^h 10y-y^2\ \mathrm{d}y$ $=\pi\left[5y^2-\frac{y^3}{3}\right]_0^h$, $=\pi\left[5h^2-\frac{h^3}{3}\right]$ $\frac{\mathrm{d}V}{\mathrm{d}h}=\pi(10h-h^2)$ Then we have $\pi\left(25-(h-5)^2\right)\times\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{\sqrt{5}\pi}{5000}\sqrt{h}$ $(10h-h^2)\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{\sqrt{5}}{5000}\sqrt{h}$ $\left(10h^{\frac{1}{2}}-h^{\frac{3}{2}}\right)\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{\sqrt{5}}{5000}$ (shown) <u>Method 2 (different expression of *V*)</u>

	$$V=\frac{500\pi}{3}-\pi\int_h^{10}25-(y-5)^2\ \mathrm{d}y$$ $$=\frac{500\pi}{3}-\pi\int_h^{10}10y-y^2\ \mathrm{d}y$$ $$=\frac{500\pi}{3}-\pi\left[5y^2-\frac{y^3}{3}\right]_h^{10}$$ $$=\frac{500\pi}{3}-\pi\left[\left(5(10)^2-\frac{(10)^3}{3}\right)-\left(5h^2-\frac{h^3}{3}\right)\right]$$ $$=\pi\left[5h^2-\frac{h^3}{3}\right]$$ $$\frac{\mathrm{d}V}{\mathrm{d}h}=\pi\left(10h-h^2\right)$$ Then we have $$\pi\left(10h-h^2\right)\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{\sqrt{5}\pi}{5000}\sqrt{h}$$ $$\left(10h^{\frac{1}{2}}-h^{\frac{3}{2}}\right)\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{\sqrt{5}}{5000}\text{ (shown)}$$
(ii)	From the DE, $$\left(10h^{\frac{1}{2}}-h^{\frac{3}{2}}\right)\frac{\mathrm{d}h}{\mathrm{d}t}=-\frac{\sqrt{5}}{5000}$$ $$\int 10h^{\frac{1}{2}}-h^{\frac{3}{2}}\ \mathrm{d}h=\int-\frac{\sqrt{5}}{5000}\ \mathrm{d}t$$ $$10\left(\frac{2}{3}h^{\frac{3}{2}}\right)-\frac{2}{5}h^{\frac{5}{2}}=-\frac{\sqrt{5}}{5000}t+B$$ When $t=0$, $h=10$: $$B=\frac{20}{3}(10)^{\frac{3}{2}}-\frac{2}{5}(10)^{\frac{5}{2}}$$ $$=84.327\text{ (to 5sf)}$$ When $h=0$: $$\frac{\sqrt{5}}{5000}t=84.327$$ $$t=188561$$ $$=189000\text{ (to 3sf)}$$

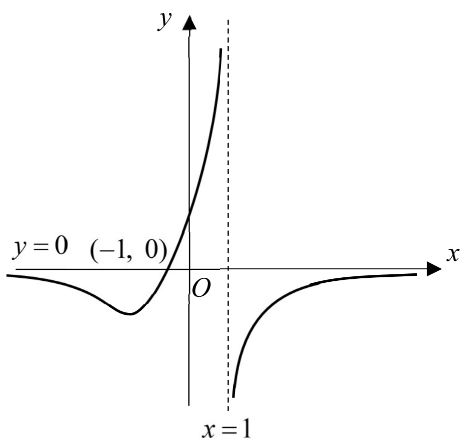
Section A: Pure Mathematics [40 marks]

1(a)	<u>Method 1: use standard expansion</u> $\mathrm{f}(x)=(4-3x)^{\frac{1}{2}}$ $=4^{\frac{1}{2}}\left(1-\frac{3x}{4}\right)^{\frac{1}{2}}$ $=2\left(1+\left(\frac{1}{2}\right)\left(-\frac{3x}{4}\right)+\frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(-\frac{3x}{4}\right)^2+\ldots\right)$ $=2-\frac{3}{4}x-\frac{9}{64}x^2+\ldots$ Validity range: $\left\|-\frac{3}{4}x\right\|<1\Rightarrow\|x\|<\frac{4}{3}\Rightarrow-\frac{4}{3}<x<\frac{4}{3}$ <u>Method 2: use repeated differentiation</u> $\mathrm{f}(x)=(4-3x)^{\frac{1}{2}}$ $\mathrm{f}'(x)=\frac{1}{2}(4-3x)^{-\frac{1}{2}}(-3)=-\frac{3}{2}(4-3x)^{-\frac{1}{2}}$ $\mathrm{f}''(x)=\left(-\frac{3}{2}\right)\left(-\frac{1}{2}\right)(4-3x)^{-\frac{3}{2}}(-3)=-\frac{9}{4}(4-3x)^{-\frac{3}{2}}$ When $x=0$, $\mathrm{f}(0)=(4-3(0))^{\frac{1}{2}}=2$ $\mathrm{f}'(0)=-\frac{3}{2}(4-3(0))^{-\frac{1}{2}}=-\frac{3}{4}$ $\mathrm{f}''(0)=-\frac{9}{4}(4-3(0))^{-\frac{3}{2}}=-\frac{9}{32}$ Then $\mathrm{f}(x)=2+\left(-\frac{3}{4}\right)x+\frac{\left(-\frac{9}{32}\right)}{2!}x^2+\ldots=2-\frac{3}{4}x-\frac{9}{64}x^2+\ldots$ Validity range: $\left\|-\frac{3}{4}x\right\|<1\Rightarrow\|x\|<\frac{4}{3}\Rightarrow-\frac{4}{3}<x<\frac{4}{3}$
(b)	Sub $x=\frac{1}{4}$ into the expansion: $\sqrt{4-3\left(\frac{1}{4}\right)}=2-\frac{3}{4}\left(\frac{1}{4}\right)-\frac{9}{64}\left(\frac{1}{4}\right)^2+\ldots=2-\frac{3}{16}-\frac{9}{1024}+\ldots$ $\sqrt{\frac{13}{4}}\approx\frac{2048-3\times64-9}{1024}=\frac{1847}{1024}$ $\sqrt{13}\approx2\times\frac{1847}{1024}=\frac{1847}{512}$ (shown)
(c)	The value obtained by substituting $x=\frac{1}{13}$ will be a better approximation for $\sqrt{13}$ because $x=\frac{1}{13}$ is closer to 0 than $x=\frac{1}{4}$.

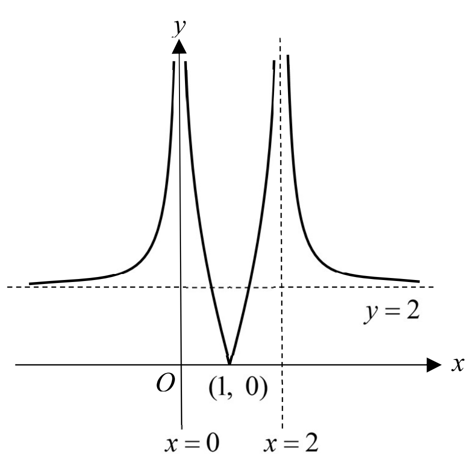
2(a)(i)



(a)(ii)



(a)(iii)



(b) $0 \le a \le 2$

3(a) $S_n = \ln\left(\frac{e^n}{3^{n^2}}\right)$

$$u_n = S_n - S_{n-1}$$
$$= \ln\left(\frac{e^n}{3^{n^2}}\right) - \ln\left(\frac{e^{n-1}}{3^{(n-1)^2}}\right)$$
$$= \ln\left(\frac{e^n}{3^{n^2}} \times \frac{3^{(n-1)^2}}{e^{n-1}}\right)$$
$$= \ln\left(\frac{3^{-2n+1}}{e^{-1}}\right)$$
$$= 1 + (1-2n)\ln 3 \quad \text{(Shown)}$$

(b) $u_n - u_{n-1} = [1 + (1-2n)\ln 3] - [1 + (1 - 2(n-1))\ln 3]$

$= -2\ln 3$ which is a constant independent of n

Hence, the sequence is an arithmetic progression.

(c) First term $= 1 - \ln 3$

Common difference $= -4\ln 3$

Sum of the first ten odd-numbered terms

$$= \frac{10}{2}[2(1-\ln 3) + 9(-4\ln 3)]$$
$$= 10 - 190\ln 3$$

(d) Method 1 – use $r = \frac{T_n}{T_{n-1}}$

$$r = \frac{e^{u_n}}{e^{u_{n-1}}}$$
$$= e^{u_n - u_{n-1}}$$
$$= e^{-2\ln 3} \quad \text{(from (b))}$$
$$= 3^{-2}$$
$$= \frac{1}{9}$$

Method 2

$$r = \frac{e^{1-3\ln 3}}{e^{1-\ln 3}} = e^{-2\ln 3} = \frac{1}{9}$$

(e) We want $\left|S_n - S_\infty\right| \le 10^{-8}$.

$$\left|\frac{a(1-r^n)}{1-r} - \frac{a}{1-r}\right| \le 10^{-8}$$

$$\Rightarrow \left|\frac{-ar^n}{1-r}\right| \le 10^{-8}$$

$$\Rightarrow \frac{e^{1-\ln 3}\left(\frac{1}{9}\right)^n}{1-\frac{1}{9}} \le 10^{-8}$$

$$\Rightarrow \frac{3e}{8}\left(\frac{1}{9}\right)^n \le 10^{-8}$$

Method 1 (algebraic)

$$n \ge \frac{\ln \frac{8(10^{-8})}{3e}}{\ln \frac{1}{9}} = 8.39233808$$

Least $n = 9$.

Method 2 (use GC)

n	$\frac{3e}{8}\left(\frac{1}{9}\right)^n$
8	2.368×10^{-8} $\quad (> 10^{-8})$
9	2.631×10^{-9} $\quad (< 10^{-8})$
10	2.923×10^{-10} $\quad (< 10^{-8})$

Least $n = 9$.

4(a) Method 1 (find p and q first)

Since $1-2\sqrt{3}\mathrm{i}$ is a root of $z^4 + pz^3 + 5z^2 + qz - 26 = 0$, then

$$\left(1-2\sqrt{3}\mathrm{i}\right)^4 + p\left(1-2\sqrt{3}\mathrm{i}\right)^3 + 5\left(1-2\sqrt{3}\mathrm{i}\right)^2 + q\left(1-2\sqrt{3}\mathrm{i}\right) - 26 = 0$$

Using G.C.,

$$73 + 152.420\mathrm{i} + p\left(-35 + 31.177\mathrm{i}\right)$$
$$+ 5\left(-11 - 6.9282\mathrm{i}\right) + q\left(1-2\sqrt{3}\mathrm{i}\right) - 26 = 0$$

Comparing real and imaginary parts,

$$73 - 35p - 55 + q - 26 = 0 \Rightarrow -35p + q = 8$$

$$152.420 + 31.177p - 34.641 - 2\sqrt{3}q = 0$$
$$\Rightarrow 31.177p - 2\sqrt{3}q = -117.779$$

Using G.C., $p = 1.00$, $q = 43.0$ (3 s.f.)

Hence, the equation is $z^4 + 1.00z^3 + 5z^2 + 43.0z - 26 = 0$

Using G.C., the roots of the equation are

$1 + 3.46\mathrm{i}$, $1 - 3.46\mathrm{i}$, -3.56 and 0.562.

<u>Method 2 (find factors first)</u>

Since the coefficients of the polynomial are real and $z = 1 - 2\sqrt{3}\mathrm{i}$ is a root, by Conjugate Root Theorem, $z = 1 + 2\sqrt{3}\mathrm{i}$ is also a root.

$$\begin{aligned} & z^4 + pz^3 + 5z^2 + qz - 26 \\ &= \left[z - \left(1 + 2\sqrt{3}\mathrm{i}\right)\right]\left[z - \left(1 - 2\sqrt{3}\mathrm{i}\right)\right]\left(z^2 + az + b\right) \\ &= \left[(z-1) - 2\sqrt{3}\mathrm{i}\right]\left[(z-1) + 2\sqrt{3}\mathrm{i}\right]\left(z^2 + az + b\right) \\ &= \left[(z-1)^2 - \left(2\sqrt{3}\mathrm{i}\right)^2\right]\left(z^2 + az + b\right) \end{aligned}$$

$$\begin{aligned} & z^4 + pz^3 + 5z^2 + qz - 26 \\ &= \left[z - \left(1 + 2\sqrt{3}\mathrm{i}\right)\right]\left[z - \left(1 - 2\sqrt{3}\mathrm{i}\right)\right]\left(z^2 + az + b\right) \\ &= \left[(z-1) - 2\sqrt{3}\mathrm{i}\right]\left[(z-1) + 2\sqrt{3}\mathrm{i}\right]\left(z^2 + az + b\right). \\ &= \left[(z-1)^2 - \left(2\sqrt{3}\mathrm{i}\right)^2\right]\left(z^2 + az + b\right) \end{aligned}$$

$$\begin{aligned} &= \left(z^2 - 2z + 1 + 12\right)\left(z^2 + az + b\right) \\ &= \left(z^2 - 2z + 13\right)\left(z^2 + az + b\right) \\ &= z^4 + z^3(-2 + a) + z^2(13 - 2a + b) + z(13a - 2b) + 13b \end{aligned}$$

Comparing the constant terms:

$13b = -26$

$b = -2$

Comparing the coefficients of z^2

$13 - 2a + b = 5$

$13 - 2a - 2 = 5$

$2a = 6$

$a = 3$

Comparing the coefficients of z^3

$p = -2 + a$

$p = 1$

Comparing the coefficients of z

$q = 13a - 2b$

$q = 39 + 4 = 43$

$\therefore$ The other roots are -3.56, 0.562 and $1 + 2\sqrt{3}\mathrm{i}$.

(b)(i) $w = \dfrac{\mathrm{i}^3}{\left(-\sqrt{3} + \mathrm{i}\right)^4}$

$$|w| = \left|\frac{\mathrm{i}^3}{\left(-\sqrt{3} + \mathrm{i}\right)^4}\right| = \frac{|\mathrm{i}|^3}{\left|\left(-\sqrt{3} + \mathrm{i}\right)\right|^4} = \frac{1^3}{\left(\sqrt{3+1}\right)^4} = \frac{1}{16}$$

$$\arg(w) = \arg\left(\frac{i^3}{\left(-\sqrt{3}+i\right)^4}\right)$$

$$= \left[3\arg(i) - 4\arg\left(-\sqrt{3}+i\right)\right]$$

$$= \left[3\left(\frac{\pi}{2}\right) - 4\left(\frac{5\pi}{6}\right)\right]$$

$$= -\frac{11\pi}{6}$$

$$\therefore \arg(w) = \frac{\pi}{6}$$

Method 2 (change to exponential form first)

$$w = \frac{i^3}{\left(-\sqrt{3}+i\right)^4} = \frac{\left(e^{\frac{\pi}{2}i}\right)^3}{\left(2e^{\frac{5\pi}{6}i}\right)^4} = \frac{e^{\frac{3\pi}{2}i}}{2^4 e^{\frac{20\pi}{6}i}}$$

$$= \frac{1}{16}e^{\frac{3\pi}{2}i - \frac{20\pi}{6}i} = \frac{1}{16}e^{-\frac{11\pi}{6}i}$$

$$|w| = \frac{1}{16}$$

$$\arg(w) = -\frac{11\pi}{6} + 2\pi = \frac{\pi}{6}$$

(b)(ii)

$$\arg\left(\frac{iw^n}{w^*}\right) = \arg(i) + n\arg(w) - \arg(w^*)$$

$$= \frac{\pi}{2} + \frac{n\pi}{6} - \left(-\frac{\pi}{6}\right)$$

$$= \frac{\pi}{2} + \frac{n\pi}{6} + \frac{\pi}{6}$$

$$\frac{\pi}{2} + \frac{n\pi}{6} + \frac{\pi}{6} = (2k+1)\left(\frac{\pi}{2}\right)$$

$$\frac{\pi}{2} + \frac{n\pi}{6} + \frac{\pi}{6} = k\pi + \frac{\pi}{2}$$

$$n + 1 = 6k$$

$$n = 6k - 1$$

Since n is positive, the smallest possible value is $n = 5$ (when $k = 1$).

Section B: Probability and Statistics [60 marks]

5(a) Let the mass in grams of a randomly chosen orange be X.

$X \sim \text{N}(150,\ 14^2)$

$\text{P}(X < 180)$
$= 0.98394$
≈ 0.984

(b) Let the mass in grams of a randomly chosen kiwi be Y.

$Y \sim \text{N}(70,\ 8^2)$

Let the mass in grams of a randomly chosen empty basket be W.

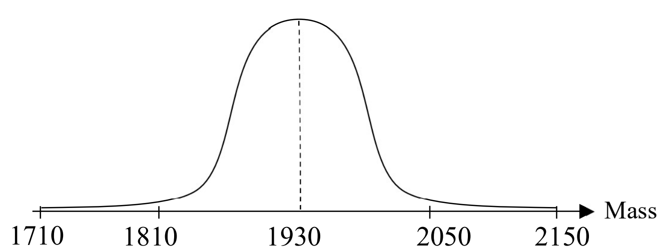
$W \sim \text{N}(750,\ 168)$

Let $M = X_1 + \ldots + X_6 + Y_1 + \ldots + Y_4 + W$

$M \sim \text{N}(1930,\ 1600)$

$\text{P}(|M - 1930| < 25)$
$= \text{P}(-25 < M - 1930 < 25)$
$= \text{P}(1905 < M < 1955)$
$= 0.46803$
≈ 0.468

(c)



$\text{P}\left(1930 - 3(40) < M < 1930 + 3\left(40\right)\right)$
$= \text{P}\left(1810 < M < 2050\right) \approx 0.997$

[Note that $\text{P}\left(M < 1810\right), \text{P}\left(M > 2050\right)$ **is very small.]**

(d) $\text{P}(M_1 > 2000) \times \text{P}(M_2 < 1900) \times \text{P}(1900 \le M_3 \le 2000) \times 3!$
$= 0.039944$
≈ 0.0399

6(a) Let X be the number of cracked screen protectors, out of 20.

$X \sim \mathrm{B}(20,\ 0.17)$

$\mathrm{P}(X \le 3) \approx 0.55041$ (5 s.f.)

(b)

X	$\mathrm{P}(X = x)$
2	0.19189
3	0.23582
4	0.20528

Most probable number of cracked screen protectors $= 3$

(c) Let Y be the number of boxes that contain no more than 3 cracked screen protectors, out of 8.

$Y \sim \mathrm{B}(8,\ 0.55041)$

$\text{Mean} = 8(0.55041) = 4.40328$

$\text{Variance} = 8(0.55041)(1-0.55041) \approx 1.9797$

Since n is large, by Central Limit Theorem,

$\overline{Y} \sim \mathrm{N}\left(4.40328,\ \frac{1.9797}{n}\right)$ approximately.

$\mathrm{P}(\overline{Y} \le 5) \ge 0.99$

n	$\mathrm{P}(\overline{Y} \le 5)$
30	$0.98991 < 0.99$
31	$0.99089 > 0.99$
32	$0.99178 > 0.99$

From GC, least n is 31.

(d)

P(ready for local sales | not ready for overseas sales)

$$= \frac{\text{P(ready for local sales AND not ready for overseas sales)}}{\text{P(not ready for overseas sales)}}$$

$$= \frac{\mathrm{P}(Y \ge 4 \ \cap\ Y < 6)}{\mathrm{P}(Y < 6)}$$

$$= \frac{\mathrm{P}(Y = 4) + \mathrm{P}(Y = 5)}{\mathrm{P}(Y \le 5)}$$

$= 0.66682$

≈ 0.667

7(a) $P(W=1)=2\left(\frac{n}{n+m+1}\times\frac{m}{n+m}\right)$

$=\frac{2nm}{(n+m+1)(n+m)}$ (shown)

$P(W=0)=\frac{m(m-1)}{(n+m+1)(n+m)}$

$P(W=2)=\frac{n(n-1)}{(n+m+1)(n+m)}$

$P(W=10)=\frac{2m}{(n+m+1)(n+m)}$

$P(W=11)=\frac{2n}{(n+m+1)(n+m)}$

(b)

$$E(W)=\frac{1}{(n+m+1)(n+m)}[2nm+2n(n-1)+20m+22n]$$
$$=\frac{1}{(n+m+1)(n+m)}\left[2nm+2n^2-2n+20m+22n\right]$$
$$=\frac{1}{(n+m+1)(n+m)}[2n(n+m)+20(n+m)]$$
$$=\frac{2(n+10)}{n+m+1}$$

For the game master to expect to make a profit,

$$E(W)<1$$
$$\frac{2(n+10)}{n+m+1}<1$$
$$2n+20<n+m+1$$
$$m-n>19 \text{ (shown)}$$

(c)

w	0	1	2	10	11
y	4	3	2	6	7
$P(W=w)$	$\frac{1560}{2550}$	$\frac{800}{2550}$	$\frac{90}{2550}$	$\frac{80}{2550}$	$\frac{20}{2550}$

Using GC,

$E(Y)=3.70196=3.70$ (3 s.f.)

$Var(Y)=0.56215=0.562$ (3 s.f.)

8(a) No. of ways

$$= ({}^9C_2\,{}^7C_2\,{}^5C_5 \times \frac{3!}{2!}) + ({}^9C_2\,{}^7C_3\,{}^4C_4 \times 3!) + ({}^9C_3\,{}^6C_3\,{}^3C_3)$$
$$= 2268 + 7560 + 1680$$
$$= 11508$$

(b) Case 1: Triangle formed from 1 point from each side $= {}^2C_1 \times {}^3C_1 \times {}^4C_1 = 24$

Case 2: Triangle formed from 2 points on XZ + 1 other $= {}^4C_2 \times {}^5C_1 = 30$

Case 3: Triangle formed from 2 points on YZ + 1 other $= {}^3C_2 \times {}^6C_1 = 18$

Case 4: Triangle formed from 2 points on XY + 1 other $= {}^2C_2 \times {}^7C_1 = 7$

Total no. of ways $= 24 + 30 + 18 + 7 = 79$

Method 2 (complement)

$$\text{No. of ways} = \begin{pmatrix}\text{no. of ways}\\ \text{of choosing 3 bulbs}\\ \text{without restrictions}\end{pmatrix} - \begin{pmatrix}\text{no. of ways}\\ \text{of choosing 3 bulbs}\\ \text{from side } YZ\end{pmatrix} - \begin{pmatrix}\text{no. of ways}\\ \text{of choosing 3 bulbs}\\ \text{from side } XZ\end{pmatrix}$$
$$= {}^9C_3 - {}^3C_3 - {}^4C_3$$
$$= 84 - 1 - 4$$
$$= 79$$

(c) $$\text{No. of ways} = \begin{pmatrix}\text{no. of ways}\\ \text{without restrictions}\end{pmatrix} - \begin{pmatrix}\text{no. of ways}\\ \text{red or blue}\\ \text{is used}\end{pmatrix} - \begin{pmatrix}\text{no. of ways}\\ \text{blue or green}\\ \text{is used}\end{pmatrix} - \begin{pmatrix}\text{no. of ways}\\ \text{red or green}\\ \text{is used}\end{pmatrix}$$
$$+ \begin{pmatrix}\text{no. of ways}\\ \text{only red}\\ \text{is used}\end{pmatrix} + \begin{pmatrix}\text{no. of ways}\\ \text{only blue}\\ \text{is used}\end{pmatrix} + \begin{pmatrix}\text{no. of ways}\\ \text{only green}\\ \text{is used}\end{pmatrix}$$
$$= 3^9 - 3(2^9) + 3$$
$$= 19683 - 1536 + 3$$
$$= 18150$$

Alternative method

$$\text{No. of ways} = \begin{pmatrix}\text{no. of ways}\\ \text{without restrictions}\end{pmatrix} - \begin{pmatrix}\text{no. of ways}\\ \text{only 1 colour}\\ \text{is used}\end{pmatrix} - \begin{pmatrix}\text{no. of ways}\\ \text{exactly 2 colours}\\ \text{are used}\end{pmatrix}$$
$$= 3^9 - 3 - {}^3C_2(2^9 - 2)$$
$$= 19683 - 3 - 1530$$
$$= 18150$$

(d) No. of ways $=(\text{Total Area})-(\text{Area 1 and 2})-(\text{Area 2 and 3})+(\text{Area 2})$

$$=\begin{pmatrix}\text{no. of ways}\\ \text{without}\\ \text{restrictions}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ 2\ A\text{'s are}\\ \text{together}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ 2\ B\text{'s are}\\ \text{together}\end{pmatrix}+\begin{pmatrix}\text{no. of ways}\\ 2\ A\text{'s \& }2\ B\text{'s}\\ \text{are together}\end{pmatrix}$$

$$=\frac{(9-1)!}{2!2!2!}-\frac{(8-1)!}{2!2!}-\frac{(8-1)!}{2!2!}+\frac{(7-1)!}{2!}$$

$$=5040-1260-1260+360$$

$$=2880$$

<u>Alternative 1 (complement)</u>

$$=(\text{Total Area})-(\text{Area 1})-(\text{Area 3})-(\text{Area 2})$$

$$=\begin{pmatrix}\text{no. of ways}\\ \text{without}\\ \text{restrictions}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ \text{A's are separate}\\ \text{B's are together}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ \text{B's are separate}\\ \text{A's are together}\end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ 2\ A\text{'s \& }2\ B\text{'s}\\ \text{are together}\end{pmatrix}$$

$$=\frac{(9-1)!}{2!2!2!}-\left(\frac{(6-1)!}{2!}\times {}^6C_2\right)-\left(\frac{(6-1)!}{2!}\times {}^6C_2\right)-\frac{(7-1)!}{2!}$$

$$=5040-900-900-360$$

$$=2880$$

<u>Alternative 2 (complement)</u>

$$=\begin{pmatrix}\text{no. of ways}\\ \text{A's are separate}\\ \end{pmatrix}-\begin{pmatrix}\text{no. of ways}\\ \text{A's are separate and}\\ \text{2B's are together}\end{pmatrix}$$

$$=\left(\frac{(7-1)!}{2!2!}\times {}^7C_2\right)-\left(\frac{(6-1)!}{2!}\times {}^6C_2\right)$$

$$=3780-900$$

$$=2880$$

<u>Alternative 3 – Use the slotting method by putting in the 2 A's first, followed by the 2 B's.</u>

No. of ways $=\begin{pmatrix}\text{no. of ways}\\ 2\ A\text{'s are separated}\\ \text{initially}\end{pmatrix}+\begin{pmatrix}\text{no. of ways}\\ 2\ A\text{'s are together}\\ \text{initially}\end{pmatrix}$

$$=\frac{(5-1)!}{2!}\times {}^5C_2\times {}^7C_2+\frac{(5-1)!}{2!}\times {}^5C_1\times {}^6C_1$$

$$=12\times 10\times 21+12\times 5\times 6$$

$$=2880$$

9(a) From GC, $\bar{x} = 3088$ (exact)

$$s^2 = 56.64509393^2 = 3208.666667 \approx 3210 \text{ (3 s.f.)}$$

(b) Since $n = 13$ is not large, we need to assume that the population fan speed is normally distributed.

Let X denote the fan speeds in RPM and μ denote the population mean.

To test $H_0 : \mu = 3100$ against $H_1 : \mu \neq 3100$ at 10% level of significance.

$$\bar{X} = \frac{X_1 + X_2 + ... + X_{13}}{13}$$

Under H_0, $\bar{X} \sim N\left(3100, \frac{30^2}{13}\right)$.

Using $\bar{X}$	Using p-value	Using z-value
Critical region: $\bar{x} < 3086.3$ or $\bar{x} > 3113.7$	For $\bar{x} = 3088$, p-value $= 0.14924$.	Critical region: $z > 1.6449$ or $z < -1.6449$ For $\bar{x} = 3088$, $z = \frac{3088 - 3100}{\sqrt{900/13}}$.
Since $\bar{x} = 3088$ does not lie in the critical region,	Since p-value > 0.10,	Since $z = -1.4422$ does not lie in the critical region,

we <u>do not reject H_0</u> and conclude that there is <u>insufficient evidence at the 10% level of significance</u> to support the claim that <u>the population mean fan speed is different from 3100 RPM</u>.

(c) The p-value of 0.14924 means that, <u>given that the population mean is indeed 3100 RPM</u>, the probability of <u>obtaining a sample mean that is less than or equal to 3088 RPM</u>, or <u>more than or equal to 3112 RPM</u>, is 0.14924.

[Mathematically, $P\left(\bar{X} \leq 3088 \cup \bar{X} \geq 3112 \mid \mu = 3100\right) = 0.14924$]

(d) From **(b)**, p-value $= 0.14924$.

Hence, least level of significance = 15%.

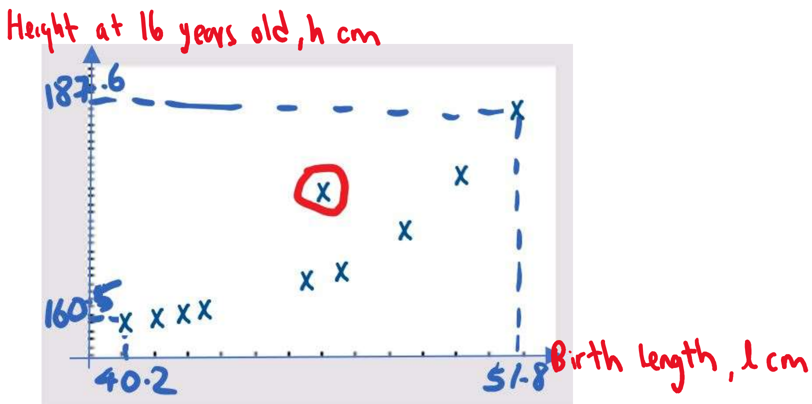
(e) Let Y denote the standby-time in hours.

To test $H_0 : \mu = 36$ against $H_1 : \mu < 36$ at 5% level of significance.

Under H_0, we have $\bar{Y} \sim N\left(36, \frac{9}{50}\right)$.

Critical region: $\bar{y} < 35.302 \Rightarrow \bar{y} < 35.3$ (3 s.f.)

10(a) Scatter diagram:



(b) From the scatter diagram, the points appear to lie on a curve rather than a straight line, so $h = a + bl$ may not model the relationship well.

(c) $h = a + bl$: $r = 0.93877$; $h = a + bl^3$: $r = 0.95886$

Since $|r|$ between h and l^3 is closer to 1 than that between l and h, this indicates a stronger linear correlation between l^3 and h, so $h = a + bl^3$ is a better model.

Equation is $h = 135.96 + 0.00033807l^3$, i.e.

$h = 136 + 0.000338l^3$ to 3 s.f.

(d) No, the regression line should not be used as there is no clear independence of either variable/ no clear independent and dependent variable.

We should use the regression line of l^3 on h to do the estimation.

(e) for $l = 46.0$, $h = 168.9$ to 1 d.p

The estimate is reliable because 46.0 is in the range $40.2 \le l \le 51.8$ so we are doing an interpolation, and r is close to 1 so there is a strong positive linear correlation between h and l^3 .

(f) Possible explanations:

- Diet
- Lifestyle
- Exercise
- Nutrition
- Halfway through growth spurt