AHMAD IBRAHIM SECONDARY SCHOOL
GCE O-LEVEL PRELIMINARY EXAMINATION 2023

SECONDARY 4 EXPRESS

Name:	Class:	Register No.:

ADDITIONAL MATHEMATICS
Paper 1

Candidates answer on the Question Paper.

4049/01
7 August 2023

2 hours 15 minutes

READ THESE INSTRUCTIONS FIRST

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The total number of marks for this paper is 90.

For Examiner's Use

/90

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
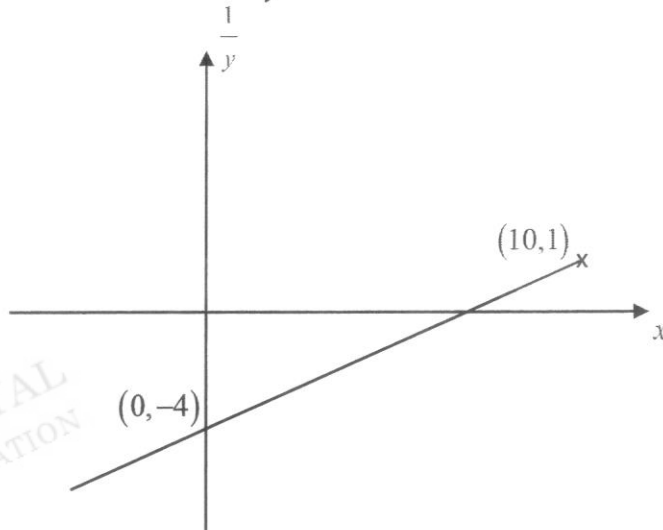$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for $\triangle ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$\Delta = \frac{1}{2}bc \sin A$$

1   The variables x and y are related by the equation $y = \dfrac{h}{2x-k}$. The diagram below shows the graph of $\dfrac{1}{y}$ against x.

Calculate the value of h and of k.   [4]

2   The Richter scale measures the intensity of an earthquake using the formula
    $M = \lg\left(\dfrac{I}{I_0}\right)$, where M is the magnitude of the earthquake, I is the intensity of
    the earthquake, and I_0 is the intensity of the smallest earthquake that can be
    measured.

   (a) Calculate the magnitude of an earthquake if its intensity is 1000 times the
        intensity of the smallest earthquake that can be measured.   [1]

   (b) In February 2011, an earthquake with magnitude 6.2 was recorded in
        Christchurch, New Zealand. Few weeks later, an earthquake with
        magnitude 9.0 was detected in Fukushima, Japan. How many times stronger
        in intensity was the Japan's earthquake as compared to the New Zealand's
        earthquake? Give your answer to 2 decimal places.   [3]

3   The diagram shows a hemispherical bowl of radius 12 cm. Water is poured into the bowl and at any time t seconds, the height of the water level from the lowest point of the hemisphere is h cm. The rate of change of the height of the water level is 0.4 cm/s.

12 cm

h cm
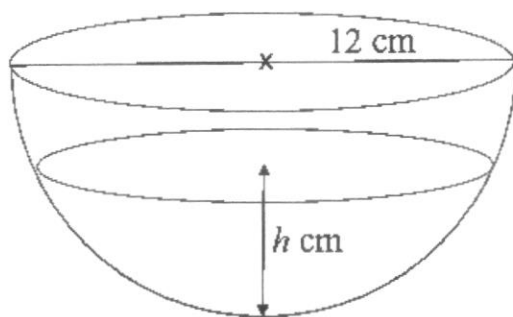
(a) Show that the area of the water surface, A, is given by $A = \pi h(24 - h)$.   [2]

(b) Find the rate of change of A when $h = 5$ cm.
Leave your answer in terms of π.   [3]

4 **(a)** Explain why there is only one solution to the equation $\log_5(13-4x) = \log_{\sqrt5}(2-x)$. [5]

(b) Solve the simultaneous equations

$$4^{x+3} = 32\left(2^{x+y}\right),$$

$$9^x + 3^y = 10.$$ [7]

5   **(a)**   Prove the identity $\cot 2x = \dfrac{1}{2\tan x} - \dfrac{1}{2}\tan x$.   [2]

(b)   Hence solve the equation $\tan x\left(3 - 4\cot 2x\right) = 3$ for $0° \le x \le 360°$.   [5]

(c)   Without further solving, explain why there are 6 roots to the equation $\tan \dfrac{x}{2}\left(3 - 4\cot x\right) = 3$ for $-360° \le x \le 720°$.   [2]

6   The curve $y = e^{2x}\sqrt{1-3x}$ intersects the y-axis at the point P. The tangent and the normal to the curve at P meet the x-axis at A and B respectively. Find the exact area of triangle PAB.   [7]
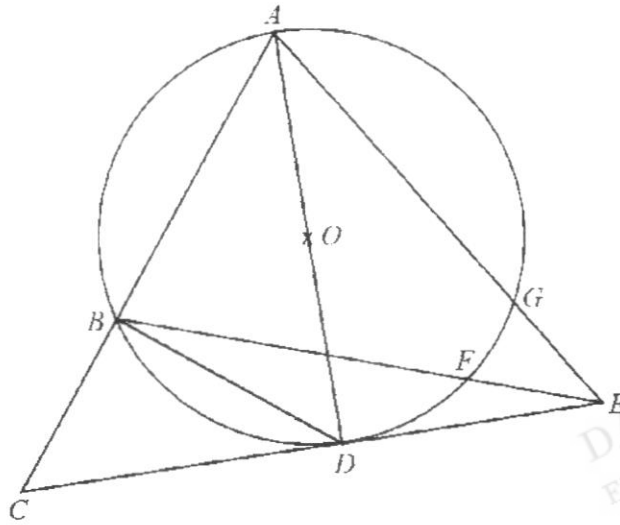
7   In the diagram, *CE* is a tangent that touches the circle of centre *O* at *D*.
    AD is the diameter of the circle, *EA* cuts the circle at points *G* and *A*, and *EB* cuts
    the circle at points *F* and *B*.

(a)   Given that *ABC* is a straight line, show that triangle *ABD* and triangle
      DBC are similar.   [3]

(b) If $BE = AE$, show that $EF = EG$. [4]

8   **(a)**   Write down the first three terms in the expansion, in ascending powers of

x, of $\left(2-\dfrac{x}{4}\right)^{n}$, where n is a positive integer greater than 2.   [3]

(b)   The first two terms in the expansion, in ascending powers of x, of

$(1+x)^{2}\left(2-\dfrac{x}{4}\right)^{n}$ are $a+bx^{2}$, where a and b are constants.

Find the value of n.   [3]
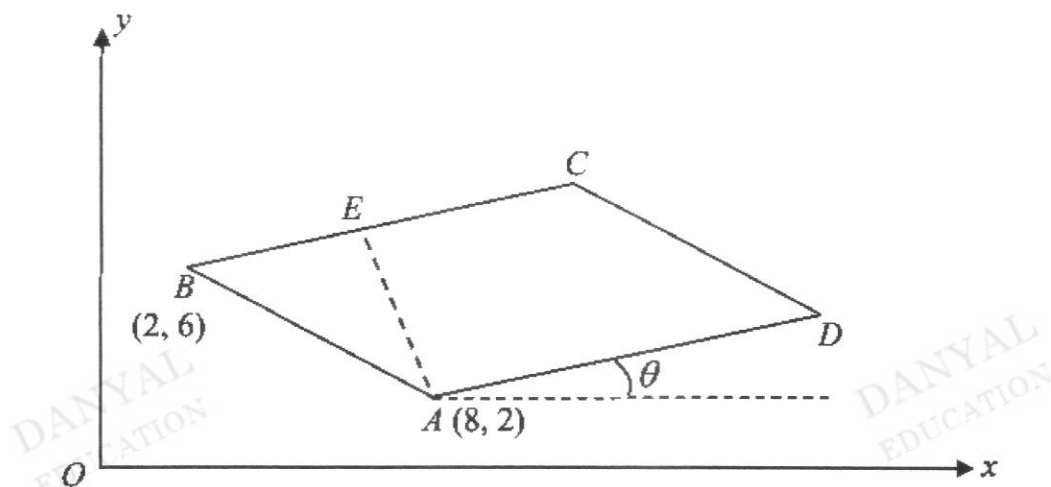
(c) Hence find the value of a and of b. [3]

9 The diagram shows a parallelogram $ABCD$ in which the coordinates of the points A and B are (8, 2) and (2, 6) respectively. The line AD makes an angle θ with the horizontal and $\tan\theta = 0.5$. The point E lies on BC such that AE is the shortest distance from A to BC.

(a) Show that the equation of line BC is $2y = x + 10$. [2]

(b) Find the equation of line AE and the coordinates of E. [3]

(c) Given that $\dfrac{BE}{BC} = \dfrac{1}{5}$, find the coordinates of C and D. [4]

(d) Find the area of the figure $OBEA$, where O is the origin. [2]

10 **(a)** Solve the equation $2\cos 3x + 1 = 0$ for $0 \le x \le \pi$. [3]

(b) Sketch the graph of $y = 2\cos 3x + 1$ for $0 \le x \le \pi$. [3]
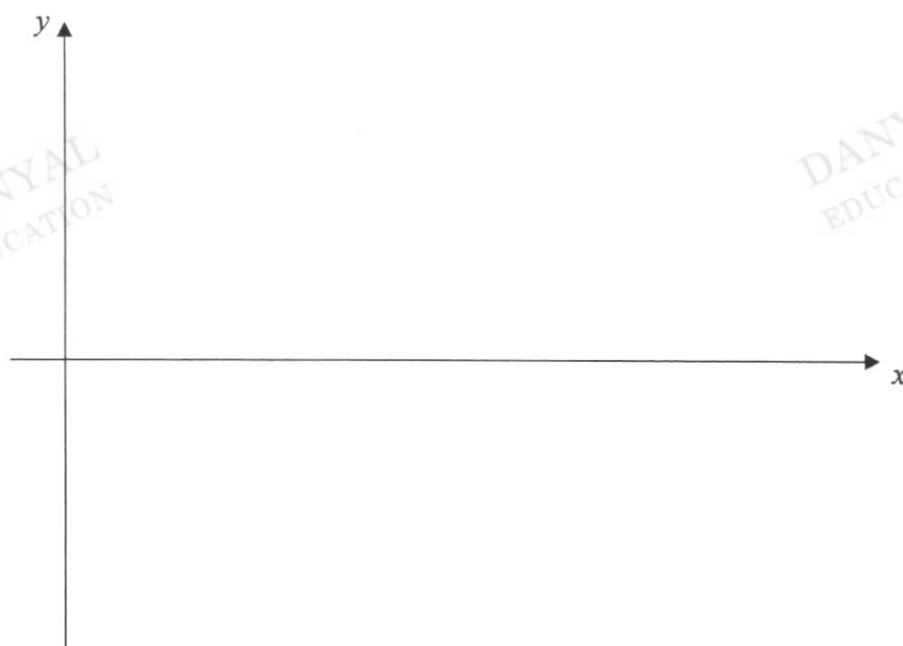
(c) The equation of a curve is $y = \dfrac{\sin 3x}{2 + \cos 3x}$, where $0 \le x \le \pi$.

Using **(a)** and **(b)**, find the range of values of x for which y is a decreasing function. [5]

11 **(a)** Express $\dfrac{3x^2+4x-20}{(2x+1)\left(x^2+4\right)}$ in partial fractions. [5]

(b) Differentiate $\ln\left(x^2+4\right)$ with respect to x. [2]

(c) The gradient function of a curve is $\dfrac{3x^2+4x-20}{(2x+1)\left(x^2+4\right)}$.

Given that the y-intercept of the curve is $(0,\ln 4)$, using part **(a)** and **(b)**, find the equation of the curve. [4]

END OF PAPER

AHMAD IBRAHIM SECONDARY SCHOOL
GCE O-LEVEL PRELIMINARY EXAMINATION 2023

SECONDARY 4 EXPRESS

Name:	Class:	Register No.:

ADDITIONAL MATHEMATICS   **4049/02**
Paper 2   **11 August 2023**

Candidates answer on the Question Paper.   **2 hours 15 minutes**

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1 **(a)** Find the range of values of x for which the expression $3-2x^2$ is negative. [2]

(b) Find the set of values of the constant k for which the curve $y=x^2$ lies entirely above the line $y=k(x+1)$. [3]

2 (a) Find the range of values of k such that the line $x+y=3$ intersects the curve $x^2-2x+2y^2=k$. [4]

(b) State a possible value of k if there is no intersection between the line and the curve. [1]

3   A polynomial, P, is $x^{2n} - (k+1)x^2 + k$ where n and k are positive integers.

   (a)   Explain why $x-1$ is a factor of P for all values of k. [2]

   (b)   Given that $k = 4$, find the value of n for which $x-2$ is a factor of P.
        Hence factorise P completely. [4]

4   A projectile was launched from a catapult to hit a defence structure on a fort. The height, h metres, of the projectile above ground is given by the equation $h = -2x^2 + 3x + 1.5$, where x metres is the horizontal distance from the catapult.

(i)   By expressing the function in the form $h = a(x-m)^2 + n$, where a, m and n are constants, explain whether the projectile can reach a height of 3 metres.   [2]

(ii) Given that the defence structure is 1.4 metres horizontally from the catapult and 0.8 metres above the ground, justify if the projectile will hit the structure.   [2]
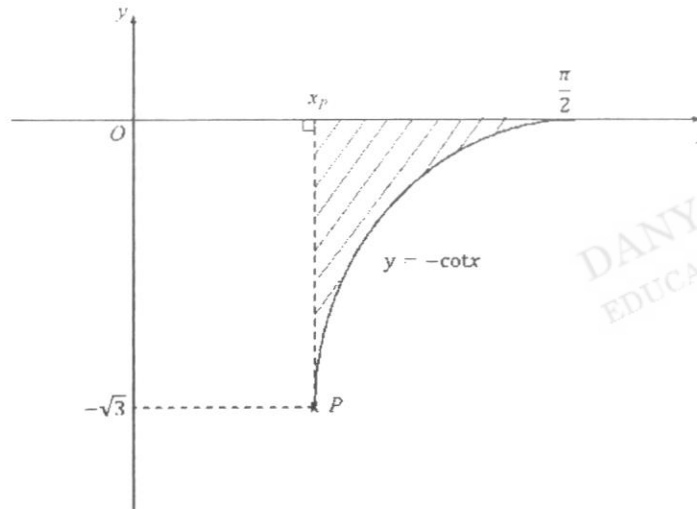
(iii) Sketch the curve of $h = -2t^2 + 3t + 1.5$. [2]

BP~29
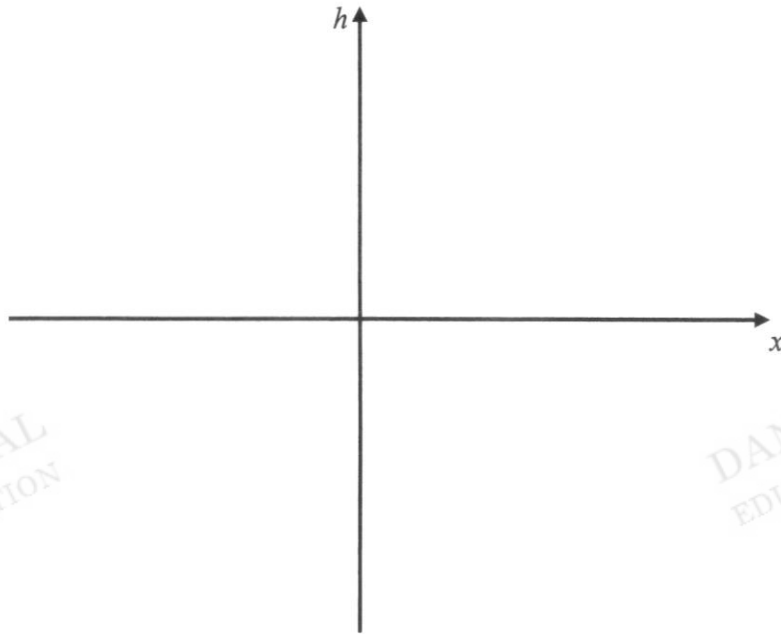
5   **(a)**   Differentiate $\ln(\sin x)$ with respect to x.   [2]

(b)

The diagram shows part of the curve $y = -\cot x$, cutting the x-axis at $\left(\dfrac{\pi}{2}, 0\right)$.

The line $y = -\sqrt{3}$ intersects the curve at P.

  (i)   State the value of x_p, the x-coordinate of P.   [1]

  (ii)   Explain why the expression $\displaystyle\int_{x_p}^{\frac{\pi}{2}} -\cot x \; dx$ does not give the area of the shaded region.   [1]

(iii) Find the exact area of the shaded region. [3]

5 (a) Without using a calculator, show that $\cos\left(\dfrac{7\pi}{12}\right) = \dfrac{1}{4}\left(\sqrt{2} - \sqrt{6}\right)$. [3]
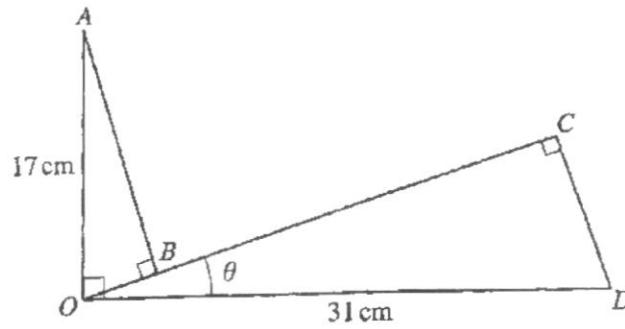
(b) Evaluate $\displaystyle\int_0^{\frac{\pi}{12}} 3\cos^2 x - \sin^2 x \ dx$ exactly. [4]

6 **(a)** **(i)** Factorise $x^6 - 64$ completely. [2]

(ii) Hence solve $x^6 - 64 = \left(x^2 + 4\right)^2 - \left(2x\right)^2$. [3]

(b) Find the values of the integers a and b for which $\dfrac{a + \sqrt{b}}{2}$ is the solution of the equation $2x\sqrt{3} + x\sqrt{125} = x\sqrt{45} + \sqrt{12}$. [4]

7

The diagram shows three fixed points O, A and D such that $OA = 17$ cm, $OD = 31$ cm and angle $AOD = 90°$.

The lines AB and DC are perpendicular to the line OC which makes an angle θ with the line OD.

The angle θ can vary in such a way that the point B lies between the points O and C.

(i) Show that $AB + BC + CD = (48\cos\theta + 14\sin\theta)$ cm. [3]

(ii) Find the values of θ for which $AB + BC + CD = 49$ cm. [6]

(iii) Find the maximum value of $AB + BC + CD$ and the corresponding value of θ. [2]

8 The diagram shows a roll of material in the shape of a cylinder of radius r cm and length l cm.

The roll is held together by three pieces of adhesive tape whose width and thickness may be ignored.

One piece of tape is in the shape of a rectangle, the other two pieces are in the shape of circles.

The total length of tape is 600 cm.
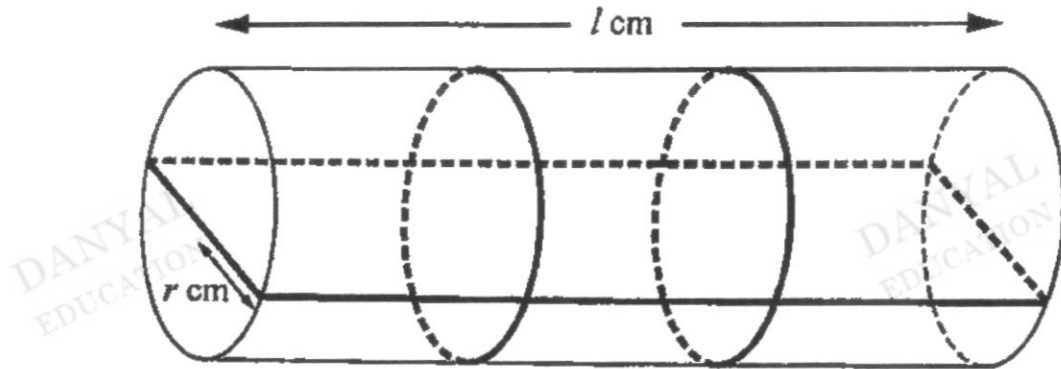
(i) Show that the volume, V cm³, of the cylinder is given by
$V = \pi r^2 \left(300 - 2r - 2\pi r\right).$ [3]

(ii) Given that r can vary, show that V has a stationary value when $r = \dfrac{k}{1+\pi}$, where k is a constant to be found, and find the corresponding value of l.   [5]

(iii) Determine if the volume is a minimum or maximum.   [3]

9   A particle travelling in a straight line passes through a fixed point O with a speed of 8 m/s.
The acceleration, a m/s^2, of the particle t s after passing through O, is given by $a = -e^{-0.1t}$.
The particle comes to instantaneous rest at the point P.

(i)   Show that the particle reaches P when $t = 10\ln 5$.   [5]

(ii)   Calculate the distance OP.   [3]

(iii) Explain why the particle is again at O at some instant during the fiftieth second after first passing through O. [3]

10
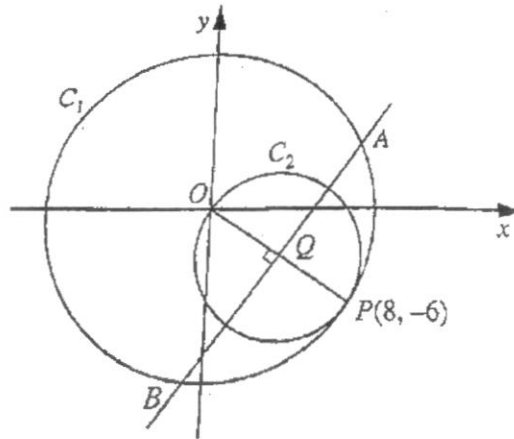
The diagram shows two circles C_1 and C_2.

Circle C_1 has its centre at the origin O.

Circle C_2 passes through O and has its centre at Q.

The point $P(8,-6)$ lies on both circles and OP is a diameter of C_2.

(a) Find the equation of C_1. [2]

(b) Explain why the equation of C_2 is $x^2 + y^2 - 8x + 6y = 0$. [3]

(c) The line through Q perpendicular to OP meets the circle C_1 at the point A and B. Show that the x-coordinates of A and B are $a + b\sqrt{3}$ and $a - b\sqrt{3}$ respectively, where a and b are integers to be found. [7]

END OF PAPER

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This document consists of **19** printed pages.

[Turn over

2

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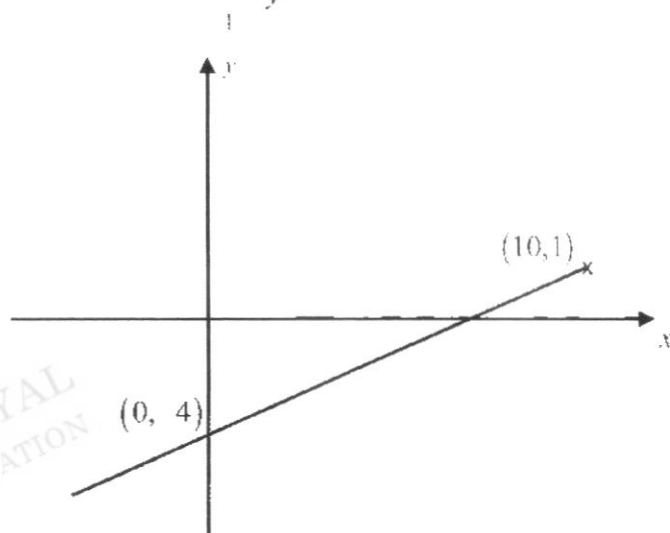$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

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3

1   The variables x and y are related by the equation $y = \dfrac{h}{2x - k}$. The diagram

     below shows the graph of $\dfrac{1}{y}$ against x.

     Calculate the value of h and of k.   [4]

$$y = \frac{h}{2x - k}$$
$$\frac{1}{y} = \frac{2x - k}{h} = \frac{2}{h}x - \frac{k}{h}$$

Gradient of line $= \dfrac{1 - (-4)}{10 - 0} = \dfrac{1}{2} = \dfrac{2}{h}$

$$h = 4$$

y-intercept at $-4 = -\dfrac{k}{h}$

$$k = 16$$

[Turn over

4

2 The Richter scale measures the intensity of an earthquake using the formula $M = \lg\left(\dfrac{I}{I_0}\right)$, where M is the magnitude of the earthquake, I is the intensity of the earthquake, and I_0 is the intensity of the smallest earthquake that can be measured.

(a) Calculate the magnitude of an earthquake if its intensity is 1000 times the intensity of the smallest earthquake that can be measured. [1]

(b) In February 2011, an earthquake with magnitude 6.2 was recorded in Christchurch, New Zealand. Few weeks later, an earthquake with magnitude 9.0 was detected in Fukushima, Japan. How many times stronger in intensity was the Japan's earthquake as compared to the New Zealand's earthquake? Give your answer to 2 decimal places. [3]

$$M = \lg\left(\frac{I}{I_0}\right)$$

$$M = \lg\left(\frac{1000 I_0}{I_0}\right) = 3$$

$$6.2 = \lg\left(\frac{I_{NZ}}{I_0}\right) \Rightarrow I_{NZ} = 10^{6.2} I_0$$

$$9.0 = \lg\left(\frac{I_J}{I_0}\right) \Rightarrow I_J = 10^{9} I_0$$

$$\frac{I_J}{I_{NZ}} = 10^{9-6.2} = 630.96 \ (2 \text{ d.p.})$$

The Japan's earthquake is 630.96 times stronger than the New Zealand's earthquake.

3   The diagram shows a hemispherical bowl of radius 12 cm. Water is poured into the bowl and at any time t seconds, the height of the water level from the lowest point of the hemisphere is h cm. The rate of change of the height of the water level is 0.4 cm/s.
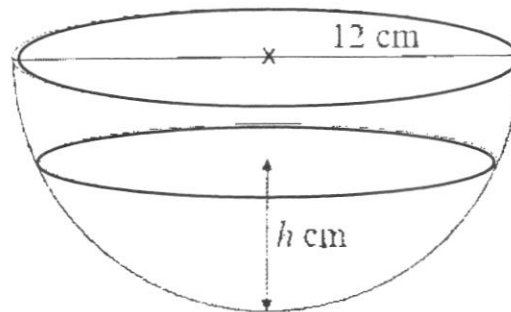
(a)   Show that the area of the water surface, A, is given by $A = \pi h(24 - h)$.   [2]

Let radius of water surface be r cm

$r^2 + (12 - h)^2 = 12^2$

$r^2 = 12^2 - (12 - h)^2$

$= (12 - (12 - h))(12 + (12 - h))$

$= h(24 - h)$

Area, $A = \pi r^2 = \pi h(24 - h)$

(b)   Find the rate of change of A when $h = 5$ cm.
Leave your answer in terms of π.   [3]

$\dfrac{dA}{dh} = \pi(24 - 2h)$

$\dfrac{dA}{dt} = \dfrac{dA}{dh} \times \dfrac{dh}{dt}$

$= \pi(24 - 2h) \times 0.4$

$= \pi(24 - 10) \times 0.4$ when $h = 5$ cm

$= 5.6\pi$

Rate of change of surface area $= 5.6\pi$ cm²/s

[Turn over

4　(a)　Explain why there is only one solution to the equation $\log_5\left(13-4x\right)=\log_{\sqrt{5}}\left(2-x\right)$. [5]

$$\log_5\left(13-4x\right)=\log_{\sqrt{5}}\left(2-x\right)$$

$$=\frac{\log_5\left(2-x\right)}{\log_5\sqrt{5}}$$

$$=\frac{\log_5\left(2-x\right)}{\frac{1}{2}}$$

$$=2\log_5\left(2-x\right)$$

$$\log_5\left(13-4x\right)=\log_5\left(2-x\right)^2$$

$$\left(13-4x\right)=\left(2-x\right)^2$$

$$\left(13-4x\right)=x^2-4x+4$$

$$x^2=9$$

$$x=3\ or\ -3$$

When $x=3$, $\log_{\sqrt{5}}\left(2-3\right)=\log_{\sqrt{5}}\left(-1\right)$ is undefined. so reject $x=3$.

The only solution is $x=-3$ (ans)

(b) Solve the simultaneous equations

$$4^{x+3} = 32\left(2^{x+y}\right),$$

$$9^x + 3^y = 10.$$ [7]

$4^{x+3} = 32\left(2^{x+y}\right)$

$2^{2x+6} = 2^5\left(2^{x+y}\right) = 2^{5+x+y}$

$2x + 6 = 5 + x + y$

$y = x + 1$ ---------- (1)

$9^x + 3^y = 10$

$3^{2x} + 3^y = 10$

$3^{2x} + 3^{x+1} = 10$

$\left(3^x\right)^2 + 3\left(3^x\right) - 10 = 0$

Let $u = 3^x$:

$u^2 + 3u - 10 = 0$

$(u + 5)(u - 2) = 0$

$u = 2$ or $u = -5$

$3^x = 2$ or $3^x = -5$ (reject since $3^x > 0$)

$x \lg 3 = \lg 2$

$x = \dfrac{\lg 2}{\lg 3} = 0.631$ (3 sf)

$y = \dfrac{\lg 2}{\lg 3} + 1 = 1.63$ (3 s.f)

[Turn over

8

5   **(a)**   Prove the identity $\cot 2x = \dfrac{1}{2\tan x} - \dfrac{1}{2}\tan x$.   [2]

$$\cot 2x = \frac{1}{2\tan x} - \frac{1}{2}\tan x$$
$$LHS = \cot 2x$$
$$= \frac{1}{\tan 2x}$$
$$= \frac{1}{\dfrac{2\tan x}{1-\tan^2 x}}$$
$$= \frac{1-\tan^2 x}{2\tan x}$$
$$= \frac{1}{2\tan x} - \frac{\tan^2 x}{2\tan x}$$
$$= \frac{1}{2\tan x} - \frac{1}{2}\tan x$$

   (b)   Hence solve the equation $\tan x\,(3 - 4\cot 2x) = 3$ for $0° \le x \le 360°$.   [5]

$$\tan x\,(3 - 4\cot 2x) = 3$$
$$\tan x\left(3 - 4\left(\frac{1}{2\tan x} - \frac{1}{2}\tan x\right)\right) = 3$$
$$\tan x\left(3 - \frac{2}{\tan x} + 2\tan x\right) = 3$$
$$3\tan x - 2 + 2\tan^2 x = 3$$
$$2\tan^2 x + 3\tan x - 5 = 0$$
$$(\tan x - 1)(2\tan x + 5) = 0$$
$$\tan x = 1 \quad \text{or} \quad \tan x = -2.5$$
$$\text{basic angle} = 45° \quad \text{or} \quad 68.199°$$
$$x = 45°, 225°, 111.8°, 291.8°$$

(c) Without further solving, explain why there are 6 roots to the equation $\tan\dfrac{x}{2}(3-4\cot x)=3$ for $360° < x < 720°$. [2]

There are 4 roots to the equation $\tan x(3-4\cot 2x)=3$ for $0° < x < 360°$ from (b).

Since the **period** of $\tan\dfrac{x}{2}(3-4\cot x)$ is **doubled** of $\tan x(3-4\cot 2x)$, there will be $4/2 = 2$ roots to the equation for $0° < x < 360°$.

For $360° < x < 720°$, the graph of $\tan\dfrac{x}{2}(3-4\cot x)$ would have **repeated 3 cycles**, thereby giving $3 \times 2 = 6$ roots to the equation.

OR

$\tan\dfrac{x}{2}(3-4\cot x)=3$, $-360° \le x \le 720°$

Let $y = \dfrac{x}{2}$, then $\tan y(3-4\cot 2y)=3$, $-180° \le y \le 360°$

y has 4 solutions in the domain $0° \le y < 360°$, **1 from each quadrant**, from (b).

Therefore, for the domain $-180° \le y \le 360°$, the graph would have entered **another half a cycle**, giving rise to 2 additional roots.

Therefore, there will be 6 solutions for y in the given domain, and thus, 6 roots to the equation.

6 The curve $y = e^{2x}\sqrt{1-3x}$ intersects the y-axis at the point P. The tangent and the normal to the curve at P meet the x-axis at A and B respectively. Find the exact area of triangle PAB. [7]

$y = e^{2x}\sqrt{1-3x}$

At P, $x = 0$: $y = 1$.

$\dfrac{dy}{dx} = 2e^{2x}\sqrt{1-3x} + e^{2x}\left(\dfrac{1}{2}\right)(1-3x)^{-1/2}(-3)$

$= 2e^{2x}\sqrt{1-3x} - \left(\dfrac{3}{2\sqrt{1-3x}}\right)e^{2x}$

$= \dfrac{e^{2x}}{\sqrt{1-3x}}\left(2(1-3x) - \dfrac{3}{2}\right)$

$= \dfrac{e^{2x}}{\sqrt{1-3x}}\left(\dfrac{1}{2} - 6x\right)$

At P, $\dfrac{dy}{dx} = \dfrac{1}{2}$.

Equation of tangent: $y - 1 = \dfrac{1}{2}(x-0)$

$y = \dfrac{1}{2}x + 1$

At A, $y = 0$: $x = -2$

Equation of normal: $y - 1 = -2(x - 0)$

$y = -2x + 1$

At B, $y = 0$: $x = \dfrac{1}{2}$

Area of triangle PAB $= \dfrac{1}{2}(1)\left(2 + \dfrac{1}{2}\right) = \dfrac{5}{4}$ units^2

7   In the diagram, CE is a tangent that touches the circle of centre O at D.
     AD is the diameter of the circle, EA cuts the circle at points G and A, and EB cuts
     the circle at points F and B.
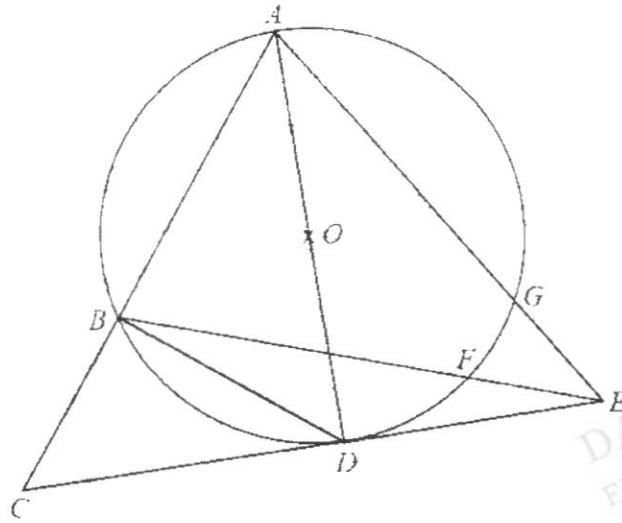
(a)   Given that ABC is a straight line, show that triangle ABD and triangle
       DBC are similar.   [3]

$\angle ABD = 90°$ (right angle in semicircle)
$\angle DBC = 90°$ (adjacent angles on a straight line)
Therefore $\angle ABD = \angle DBC$.

$\angle CDB = \angle DAB$ (angles in alt. segment)

Since there are 2 pairs of corresponding angles that are equal, triangle ABD and
triangle DBC are similar.

12

(b) If $BE = AE$, show that $EF = EG$. [4]

$$\angle EFG = 180° - \angle BFG \quad \text{(adj angles on a str. line)}$$
$$= 180° - (180° - \angle BAG) \quad \text{(angles in opp. segment)}$$
$$= \angle BAG$$
$$= \angle EAB$$

$$\angle EGF = 180° - \angle FGA \quad \text{(adj angles on a str. line)}$$
$$= 180° - (180° - \angle FBA) \quad \text{(angles in opp. segment)}$$
$$= \angle FBA$$
$$= \angle EBA$$

Given $BE = AE$, triangle EAB is isosceles.
$\angle EBA = \angle EAB$ (base angle of isos. triangle)

Therefore $\angle EGF = \angle EFG$
Hence triangle EGF is isosceles, and $EF = EG$

8 (a) Write down the first three terms in the expansion, in ascending powers of x, of $\left(2-\dfrac{x}{4}\right)^{n}$, where n is a positive integer greater than 2. [3]

$$\left(2-\frac{x}{4}\right)^{n}=2^{n}+\binom{n}{1}(2)^{n-1}\left(-\frac{x}{4}\right)+\binom{n}{2}(2)^{n-2}\left(-\frac{x}{4}\right)^{2}+\ldots$$

$$=2^{n}-\frac{1}{8}nx(2^{n})+\frac{n(n-1)}{32}(2^{n})x^{2}+\ldots$$

$$=2^{n}-nx(2^{n-3})+n(n-1)(2^{n-2})x^{2}+\ldots$$

(b) The first two terms in the expansion, in ascending powers of x, of $(1+x)^{2}\left(2-\dfrac{x}{4}\right)^{n}$ are $a+bx^{2}$, where a and b are constants. Find the value of n. [3]

$$(1+x)^{2}\left(2-\frac{x}{4}\right)^{n}=(1+2x+x^{2})\left(2^{n}-nx(2^{n-3})+n(n-1)(2^{n-5})x^{2}+\ldots\right)$$

$$=2^{n}-nx(2^{n-3})+n(n-1)(2^{n-5})x^{2}+2^{n+1}x-nx^{2}(2^{n-2})+(2^{n})x^{2}$$

$$2^{n}+x\left(2^{n+1}-n(2^{n-3})\right)+x^{2}\left[n(n-1)(2^{n-5})-n(2^{n-2})+(2^{n})\right]$$

$$=a+bx^{2}$$

Comparing coefficient of x: $2^{n+1}-n(2^{n-3})=0$

$$2^{n-3}(2^{4}-n)=0$$

Since $2^{n-3}>0$ for all real values of n, $n=2^{4}=16$

(c) Hence find the value of a and of b. [3]

Comparing constant: $a=2^{16}=65536$

Comparing coefficient of x^{2} :
$$b=16(15)(2^{16-5})-16(2^{16-2})+(2^{16})=294912$$

9   The diagram shows a parallelogram $ABCD$ in which the coordinates of the points A and B are (8, 2) and (2, 6) respectively. The line AD makes an angle θ with the horizontal and $\tan\theta = 0.5$. The point E lies on BC such that AE is the shortest distance from A to BC.
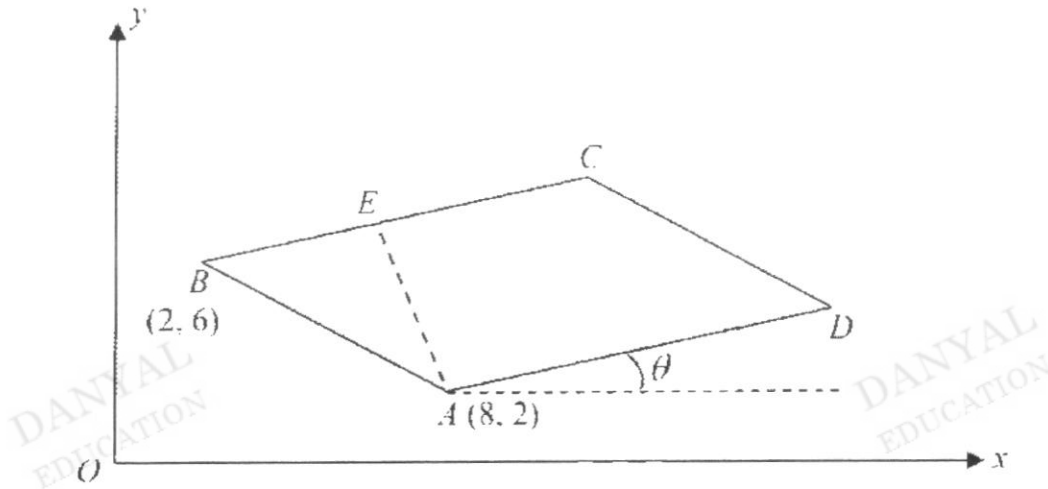
(a)   Show that the equation of line BC is $2y = x + 10$.   [2]

Gradient of BC = Gradient of AD ($BC // AD$)

$$\tan\theta = \frac{1}{2}$$

Equation of BC: $y - 6 = \frac{1}{2}(x - 2)$

$$y = \frac{1}{2}x - 1 + 6$$

$$y = \frac{1}{2}x + 5$$

$$2y = x + 10$$

(b)   Find the equation of line AE and the coordinates of E.   [3]

Gradient of AE = -2

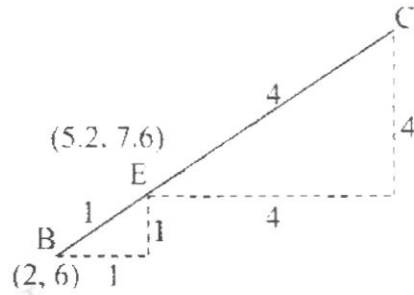Equation of AE: $y - 2 = -2(x - 8)$

$$y = -2x + 18$$

At intersection: $\frac{1}{2}x + 5 = -2x + 18$

$x = 5.2$, $y = 7.6$

Coordinates of E: (5.2, 7.6)

(c) Given that $\dfrac{BE}{BC} = \dfrac{1}{5}$, find the coordinates of C and D. [4]

Let coordinates of C be (x, y).
Using similar triangles,

$$\frac{1}{4} = \frac{5.2 - 2}{x - 5.2} = \frac{7.6 - 6}{y - 7.6}$$

$$x = 4(3.2) + 5.2 = 18$$

$$y = 4(1.6) + 7.6 = 14$$

Coor of C: (18, 14)

Midpoint of AC: (13, 8)   Midpoint of BD (p, q)

$$\left(\frac{2 + p}{2}, \frac{6 + q}{2} \right) = (13, 8)$$

$$p = 24, \ q = 10$$

Coordinates of D (24, 10)

(d) Find the area of the figure $OBEA$, where O is the origin. [2]

$$\text{Area} = \frac{1}{2} \begin{vmatrix} 0 & 8 & 5.2 & 2 & 0 \\ 0 & 2 & 7.6 & 6 & 0 \end{vmatrix}$$

$$\frac{1}{2}(60.8 + 31.2 - 10.4 - 15.2) = 33.2 \text{ units}^2$$

16

10 **(a)** Solve the equation $2\cos 3x + 1 = 0$ for $0 \le x \le \pi$. [3]
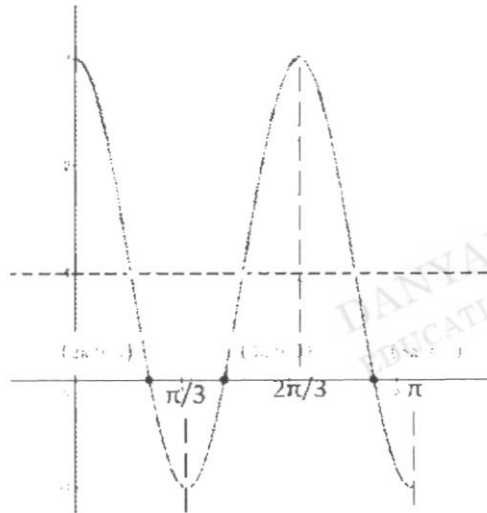
$2\cos 3x + 1 = 0$

$\cos 3x = -\dfrac{1}{2}$

Basic angle $= \cos^{-1}\left(\dfrac{1}{2}\right) = \dfrac{\pi}{3}$

Since $0 < x < \pi$, $0 < 3x < 3\pi$

$3x = \dfrac{2\pi}{3}, \dfrac{4\pi}{3}, \dfrac{8\pi}{3}$

$x = \dfrac{2\pi}{9}, \dfrac{4\pi}{9}, \dfrac{8\pi}{9}$

(b) Sketch the graph of $y = 2\cos 3x + 1$ for $0 \le x < \pi$. [3]

(c) The equation of a curve is $y = \dfrac{\sin 3x}{2 + \cos 3x}$, where $0 \le x \le \pi$.

Using (a) and (b), find the range of values of x for which y is a decreasing function. [5]

$$y = \frac{\sin 3x}{2 + \cos 3x}$$

$$\frac{dy}{dx} = \frac{(2 + \cos 3x)3\cos 3x - \sin 3x(-3\sin 3x)}{(2 + \cos 3x)^2}$$

$$= \frac{6\cos 3x + 3\cos^2 3x + 3\sin^2 3x}{(2 + \cos 3x)^2}$$

$$= \frac{6\cos 3x + 3}{(2 + \cos 3x)^2}$$

For decreasing function, $\dfrac{dy}{dx} < 0$

$$\frac{6\cos 3x + 3}{(2 + \cos 3x)^2} < 0$$

Since $(2 + \cos x)^2 > 0$ for all values of x, $6\cos 3x + 3 < 0$

$$2\cos 3x + 1 < 0$$

$$\frac{2\pi}{9} < x < \frac{4\pi}{9} \quad \text{or} \quad \frac{8\pi}{9} < x < \pi$$

11 (a) Express $\dfrac{3x^2 + 4x - 20}{(2x+1)(x^2+4)}$ in partial fractions. [5]

$$\frac{3x^2 + 4x - 20}{(2x+1)(x^2+4)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+4}$$

$$3x^2 + 4x - 20 = A(x^2 + 4) + (Bx + C)(2x+1)$$

Let $x = -\dfrac{1}{2}$ $\quad 3\left(-\dfrac{1}{2}\right)^2 + 4\left(-\dfrac{1}{2}\right) - 20 = A\left(\left(-\dfrac{1}{2}\right)^2 + 4\right) + 0$

$$A = -5$$

Let $x = 0$ $\qquad -20 = A(4) + C(1)$

$$C = 0$$

Let $x = 1$ $\qquad -13 = 5A + 3(B + C)$

$$B = 4$$

$$\frac{3x^2 + 4x - 20}{(2x+1)(x^2+4)} = \frac{4x}{x^2+4} - \frac{5}{2x+1}$$

(b) Differentiate $\ln(x^2 + 4)$ with respect to x. [2]

$$\frac{d}{dx} \ln(x^2 + 4) = \frac{2x}{x^2+4}$$

(c) The gradient function of a curve is $\dfrac{3x^2 + 4x - 20}{(2x+1)(x^2+4)}$.

Given that the y-intercept of the curve is $(0, \ln 4)$, using part (a) and (b), find the equation of the curve. [4]

$$\frac{dy}{dx} = \frac{3x^2 + 4x - 20}{(2x+1)(x^2+4)} = \frac{4x}{x^2+4} - \frac{5}{2x+1}$$

$$y = \int \frac{4x}{x^2+4} - \frac{5}{2x+1} \, dx$$

$$= \int \frac{4x}{x^2+4} \, dx - \int \frac{5}{2x+1} \, dx$$

$$= 2\int \frac{2x}{x^2+4} \, dx - 5\int \frac{1}{2x+1} \, dx$$

$$= 2\ln(x^2+4) - \frac{5}{2}\ln(2x+1) + C$$

Since $(0, \ln 4)$ is on the curve: $\ln 4 = 2\ln(4) - \frac{5}{2}\ln(1) + C$

$$C = -\ln 4$$

$$y = 2\ln(x^2+4) - \frac{5}{2}\ln(2x+1) - \ln 4$$

END OF PAPER

AHMAD IBRAHIM SECONDARY SCHOOL
GCE O-LEVEL PRELIMINARY EXAMINATION 2023

SECONDARY 4 EXPRESS

Name:	Class:	Register No.:
MARKING SCHEME		

ADDITIONAL MATHEMATICS
Paper 2

4049/02
11 August 2023

Candidates answer on the Question Paper.

2 hours 15 minutes

READ THESE INSTRUCTIONS FIRST

Write your name, class and index number on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** questions.
Give non-exact numerical answers to 3 significant figures, or 1 decimal place in the case
of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved scientific calculator is expected, where appropriate.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 90.

For Examiner's Use

/90

This document consists of **20** printed pages.

2

Mathematical Formulae

1. ALGEBRA

Quadratic Equation

For the equation $ax^2 + bx + c = 0$,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n$$

where n is a positive integer and $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

2. TRIGONOMETRY

Identities

$$\sin^2 A + \cos^2 A = 1$$
$$\sec^2 A = 1 + \tan^2 A$$
$$\operatorname{cosec}^2 A = 1 + \cot^2 A$$
$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$
$$\tan(A \pm B) = \frac{\tan A + \tan B}{1 \mp \tan A \tan B}$$
$$\sin 2A = 2 \sin A \cos A$$
$$\cos 2A = \cos^2 A - \sin^2 A = 2\cos^2 A - 1 = 1 - 2\sin^2 A$$
$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Formulae for $\triangle ABC$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$\Delta = \frac{1}{2} bc \sin A$$

1 (a) Find the range of values of x for which the expression $3 - 2x^2$ is negative.   [2]

$$3 - 2x^2 < 0$$
$$2x^2 - 3 > 0$$
$$x < \sqrt{\frac{3}{2}} \quad \text{or} \quad x > \sqrt{\frac{3}{2}}$$
$$x < \frac{\sqrt{6}}{2} \quad \text{or} \quad x > \frac{\sqrt{6}}{2}$$

(b) Find the set of values of the constant k for which the curve $y = x^2$ lies entirely above the line $y = k(x+1)$.   [3]

$$x^2 > k(x+1)$$
$$x^2 - kx - k > 0$$

For the quadratic expression to be always positive.

$$b^2 - 4ac < 0$$
$$k^2 + 4k < 0$$
$$k(k+4) < 0$$
$$-4 < k < 0$$

2 **(a)** Find the range of values of k such that the line $x + y = 3$ intersects the curve $x^2 - 2x + 2y^2 = k$. [4]

$x + y = 3$

$y = 3 - x \text{-------}(1)$

$x^2 - 2x + 2(3-x)^2 = k$

$x^2 - 2x + 2(9 - 6x + x^2) = k$

$x^2 - 2x + 18 - 12x + 2x^2 - k = 0$

$3x^2 - 14x + 18 - k = 0$

Since the line and curve intersects,

$b^2 - 4ac \geq 0$

$(-14)^2 - 4(3)(18-k) \geq 0$

$196 - 216 + 12k \geq 0$

$-20 + 12k \geq 0$

$k \geq \dfrac{5}{3}$

(b) State a possible value of k if if there is no intersection between the line and the curve. [1]

Any value that is $< \dfrac{5}{3}$.

3 A polynomial, P, is $x^{2n} - (k+1)x^2 + k$ where n and k are positive integers.

(a) Explain why $x - 1$ is a factor of P for all values of k. [2]

let $f(x) = x^{2n} - (k+1)x^2 + k$

$f(1) = 1 - k - 1 + k$

$\quad = 0$

$\therefore$ since remainder $= 0$, $(x - 1)$ is a factor.

(b) Given that $k = 4$, find the value of n for which $x - 2$ is a factor of P.
Hence factorise P completely. [4]

$f(x) = x^{2n} - 5x^2 + 4$

$f(2) = 2^{2n} - 16$

Since $x - 2$ is a factor,

$2^{2n} - 16 = 0$

$n = 2$

$f(x) = x^4 - 5x^2 + 4$

$\quad = (x-1)(x-2)(x^2 + 3x + 2)$

$\quad = (x-1)(x-2)(x+1)(x+2)$

6

4 A projectile was launched from a catapult to hit a defence structure on a fort. The height, h metres, of the projectile above ground is given by the equation $h = -2x^2 + 3x + 1.5$, where x metres is the horizontal distance from the catapult.

(i) By expressing the function in the form $h = a(x-m)^2 + n$, where a, m and n are constants, explain whether the projectile can reach a height of 3 metres. [2]

$h = -2x^2 + 3x + 1.5$

$= -2(x^2 - 1.5x - 0.75)$

$-2[(x-0.75)^2 - 0.75^2 - 0.75]$

$-2(x-0.75)^2 + 2.625$

maximum point is $(0.75, 2.625)$

Therefore the projectile cannot reach a height of 3m

since the maximum height is 2.625m.

(ii) Given that the defence structure is 1.4 metres horizontally from the catapult and 0.8 metres above the ground, justify if the projectile will hit the structure. [2]

Subs $(1.4, 0.8)$ into $h = -2x^2 + 3x + 1.5$
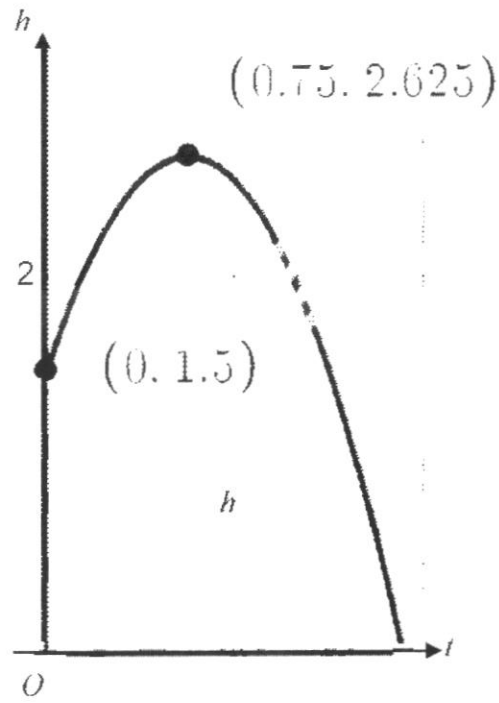
$h = -2(1.4)^2 + 3(1.4) + 1.5$

1.78

$\neq 0.8$

Since the point $(1.4, 0.8)$ does not lie on the curve $h = -2x^2 + 3x + 1.5$,

therefore the projectile will not hit the structure.
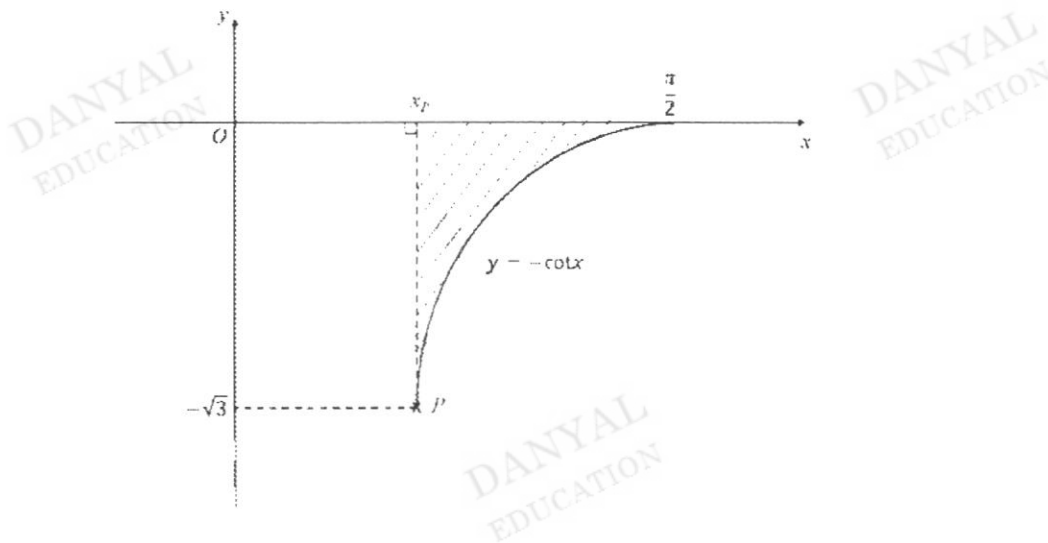
(iii) Sketch the curve of $h = 2x^2 + 3x + 1.5$. [2]

$(0.75, 2.625)$

2

$(0, 1.5)$

h

5 (a) Differentiate $\ln(\sin x)$ with respect to x.   [2]

$$\frac{d}{dx}\ln(\sin x)$$

$$\frac{1}{\sin x}\cos x$$

$$\cot x$$

(b)

The diagram shows part of the curve $y = -\cot x$, cutting the x-axis at $\left(\frac{\pi}{2}, 0\right)$.

The line $y = -\sqrt{3}$ intersects the curve at P.

(i) State the value of x_p, the x-coordinate of P.   [1]

$$\sqrt{3} = \frac{1}{\tan x}$$

$$\tan x \quad \frac{1}{\sqrt{3}}$$

$$x \quad \frac{\pi}{6}$$

(ii) Explain why the expression $\int_{x_p}^{\pi} -\cot x \, dx$ does not give the area of the shaded region. [1]

The shaded area is below the x-axis. If we $\int_{Q}^{\pi} \cot x \, dx$, we will get a **negative**

value for the area. Thus $\int_{Q}^{\pi} -\cot x \, dx$ does not give area of the shaded region.

(iii) Find the exact area of the shaded region. [3]

$$y = -\frac{1}{\tan x}$$

when $y = -\sqrt{3}$

$$\int_{\frac{\pi}{6}}^{\pi} -\cot x \, dx$$

$$= [\ln(\sin x)]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$$

$$= \ln 1 - \ln \frac{1}{2}$$

$$= \ln 2 \text{ units}^2 \text{ or } \ln \frac{1}{2} \text{ units}^2$$

6 (a) Without using a calculator, show that $\cos\left(\dfrac{7\pi}{12}\right) = \dfrac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$. [3]

$$\cos\left(\frac{7\pi}{12}\right) = \cos\left(\frac{\pi}{4}+\frac{\pi}{3}\right)$$

$$= \cos\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{3}\right) - \sin\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{\sqrt{2}}{2}\left(\frac{1}{2}\right) - \frac{\sqrt{2}}{2}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{4}\left(\sqrt{2}-\sqrt{6}\right)$$

(b) Evaluate $\displaystyle\int_{0}^{\frac{\pi}{12}} 3\cos^2 x - \sin^2 x \; dx$ exactly. [4]

$$\int_{0}^{\frac{\pi}{12}} 3\cos^2 x - \sin^2 x \; dx = \int_{0}^{\frac{\pi}{12}} \frac{3}{2}(2\cos^2 x - 1 + 1) + \frac{1}{2}(1 - 2\sin^2 x - 1) \; dx$$

$$= \int_{0}^{\frac{\pi}{12}} \frac{3}{2}\cos 2x + \frac{1}{2}\cos 2x + 1 \; dx$$

$$= \int_{0}^{\frac{\pi}{12}} 2\cos 2x + 1 \; dx$$

$$= \left[\sin 2x + x\right]_{0}^{\frac{\pi}{12}}$$

$$= \sin\frac{\pi}{6} + \frac{\pi}{12}$$

$$\frac{6 + \pi}{12}$$

7 (a) (i) Factorise $x^6 - 64$ completely. [2]

$x^6 - 64$
$= \left(x^2\right)^3 - 4^3$
$= \left(x^2 - 4\right)\left(x^4 + 4x^2 + 16\right)$
$= (x-2)(x+2)\left(x^4 + 4x^2 + 16\right)$

OR

$\left(x^3\right)^2 - \left(2^3\right)^2$
$= \left(x^3 - 8\right)\left(x^3 + 8\right)$
$= \left(x^3 - 2^3\right)\left(x^3 + 2^3\right)$
$(x-2)\left(x^2 + 2x + 4\right)(x+2)\left(x^2 - 2x + 4\right)$
$= (x-2)(x+2)\left(x^2 + 2x + 4\right)\left(x^2 - 2x + 4\right)$

(ii) Hence solve $x^6 - 64 = \left(x^2 + 4\right)^2 - (2x)^2$. [3]

$x^6 - 64 = \left(x^2 + 4\right)^2 - (2x)^2$
$(x+2)(x-2)\left(x^4 + 4x^2 + 16\right) = x^4 + 8x^2 - 16 - 4x^2$
$(x+2)(x-2)\left(x^4 + 4x^2 + 16\right) = x^4 + 4x^2 + 16$
$(x+2)(x-2) = 1$
$x^2 - 5 = 0$
$x = \pm\sqrt{5}$

(b) Find the values of the integers a and b for which $\dfrac{a+\sqrt{b}}{2}$ is the solution of the equation

$2x\sqrt{3}+x\sqrt{125}=x\sqrt{45}+\sqrt{12}.$ [4]

$2x\sqrt{3}+x\sqrt{125}=x\sqrt{45}+\sqrt{12}$

$2x\sqrt{3}+5x\sqrt{5}=3x\sqrt{5}+2\sqrt{3}$

$2x\sqrt{3}+5x\sqrt{5}=3x\sqrt{5}-2\sqrt{3}$
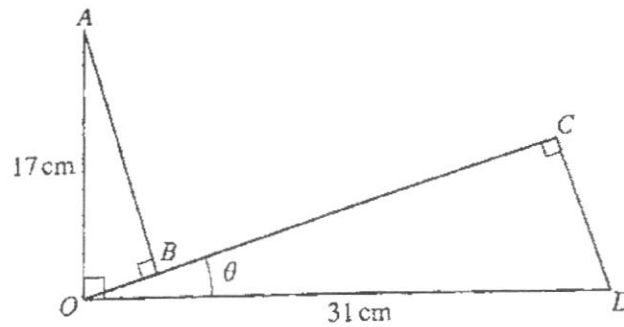
$x\left(2\sqrt{3}+2\sqrt{5}\right)=2\sqrt{3}$

$x=\dfrac{2\sqrt{3}}{\left(2\sqrt{3}+2\sqrt{5}\right)}\times\dfrac{\left(2\sqrt{3}-2\sqrt{5}\right)}{\left(2\sqrt{3}-2\sqrt{5}\right)}$

$\dfrac{12-4\sqrt{15}}{12-20}$

$=\dfrac{12-4\sqrt{15}}{8}$

$=\dfrac{-3+\sqrt{15}}{2}$

$a=-3,\ b=15$

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The diagram shows three fixed points O, A and D such that $OA = 17$ cm, $OD = 31$ cm and angle $AOD = 90°$.
The lines AB and DC are perpendicular to the line OC which makes an angle θ with the line OD.
The angle θ can vary in such a way that the point B lies between the points O and C.

(i)   Show that $AB + BC + CD = (48\cos\theta + 14\sin\theta)$ cm.   [3]

$$\sin\theta = \frac{CD}{31}$$

$$CD = 31\sin\theta$$

$$\sin\theta = \frac{OB}{17}$$

$$OB = 17\sin\theta$$

$$\cos\theta = \frac{AB}{17}$$

$$AB = 17\cos\theta$$

$$\cos\theta = \frac{OC}{31}$$

$$OC = 31\cos\theta$$

$$AB + BC + CD$$

$$= 17\cos\theta + 31\cos\theta - 17\sin\theta + 31\sin\theta$$

$$= (48\cos\theta + 14\sin\theta) \text{ cm}$$

(ii) Find the values of θ for which $AB + BC + CD = 49 \text{ cm}$. [6]

$$48\cos\theta + 14\sin\theta = 49$$
$$R\cos(\theta - \alpha) = 49$$
$$R = \sqrt{48^2 + 14^2}$$
$$= 50$$
$$\alpha = \tan^{-1}\left(\frac{14}{48}\right)$$
$$= 16.26°$$
$$50\cos(\theta - 16.26°) = 49$$
$$\cos(\theta - 16.26°) = \frac{49}{50}$$
Reference angle $= 11.48°$
$$\theta = 27.7°, 4.8°$$

(iii) Find the maximum value of $AB + BC + CD$ and the corresponding value of θ. [2]
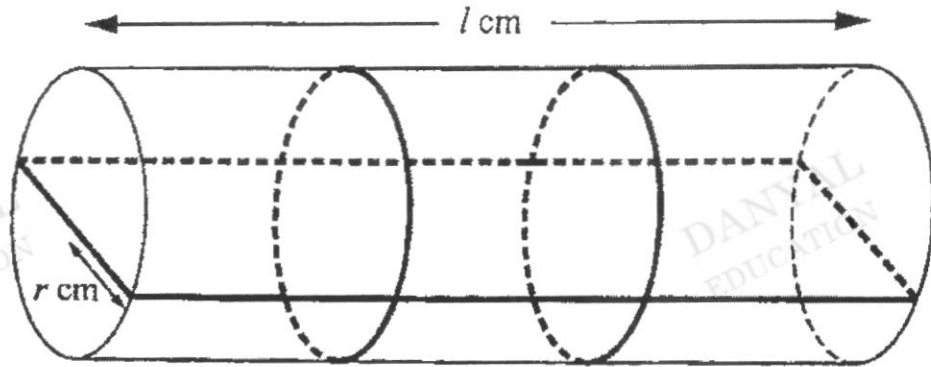
$$\text{max } 50\cos(\theta - 16.26°)$$
$$= 50$$
occurs when $\cos(\theta - 16.26°) = 1$
$$\theta - 16.26° = 0$$
$$\theta = 16.3° \text{ (1 d.p)}$$

9   The diagram shows a roll of material in the shape of a cylinder of radius r cm and length l cm.

The roll is held together by three pieces of adhesive tape whose width and thickness may be ignored.

One piece of tape is in the shape of a rectangle, the other two pieces are in the shape of circles.

The total length of tape is 600 cm.

(i)   Show that the volume, V cm^3, of the cylinder is given by
$V = \pi r^2 (300 - 2r - 2\pi r)$.   [3]

$$2(2r) + 2(l) + 2(2\pi r) = 600$$
$$2r + l + 2\pi r = 300$$
$$l = 300 - 2r - 2\pi r$$
$$V = \pi r^2 l$$
$$= \pi r^2 (300 - 2r - 2\pi r) \quad \text{(shown)}$$

(ii) Given that r can vary, show that V has a stationary value when $r = \dfrac{k}{1+\pi}$, where k is a constant to be found, and find the corresponding value of V. [5]

$$\frac{dV}{dr} = 600\pi r - 6\pi r^2 - 6\pi^2 r^2$$
$$= 6\pi r(100 - r - \pi r)$$

For stationary value, $\dfrac{dV}{dr} = 0$.

$6\pi r(100 - r - \pi r) = 0$

$r = 0$ rejected because $r > 0$ or

$100 - r - \pi r = 0$

$(1+\pi)r - 100$

$$r = \frac{100}{1+\pi}$$

$\therefore V$ has a stationary value when $r = \dfrac{100}{1+\pi}$, where $k = 100$. (shown)

When $r = \dfrac{100}{1+\pi}$,

$$V = 300 - 2\left(\frac{100}{1+\pi}\right) - 2\pi\left(\frac{100}{1+\pi}\right)$$
$$= 300 - 2\left(\frac{100}{1+\pi}\right)(1+\pi)$$
$$= 300 - 200$$
$$= 100$$

(iii) Determine if the volume is a minimum or maximum. [3]

$$\frac{dV}{dr} = 6\pi r(100 - r - \pi r)$$

$$\frac{d^2V}{dr^2} = 6\pi r(-1 - \pi) + (100 - r - \pi r)(6\pi)$$

$$= 6\pi r(-1 - \pi) + (100 - r - \pi r)(6\pi)$$

$$= 6\pi(-r - \pi r + 100 - r - \pi r)$$

$$= 6\pi(-2r - 2\pi r + 100)$$

when $r = \dfrac{100}{1+\pi}$,

$$\frac{d^2V}{dr^2} = 6\pi\left(-\frac{200}{1+\pi} - \frac{200\pi}{1+\pi} + 100\right)$$

$$-600\pi$$

Since $\dfrac{d^2V}{dr^2} < 0$, V is a maximum.

10 A particle travelling in a straight line passes through a fixed point O with a speed of 8 m/s.

The acceleration, a m/s^2, of the particle t s after passing through O, is given by $a = -e^{0.1t}$. The particle comes to instantaneous rest at the point P.

(i) Show that the particle reaches P when $t = 10\ln 5$. [5]

$$a = -e^{0.1t}$$

$$v = \int a = -e^{-0.1t} \, dt$$

$$= \frac{e^{-0.1t}}{-0.1} + c$$

When $t = 0$, $v = 8$

$$8 = \frac{-1}{-0.1} + c$$

$$8 = 10 + c$$

$$c = -2$$

$$v = 10e^{0.1t} - 2$$

when $v = 0$

$$10e^{-0.1t} - 2 = 0$$

$$e^{-0.1t} = \frac{1}{5}$$

$$-0.1t = -\ln 5$$

$$t = 10\ln 5 \text{ (shown)}$$

(ii) Calculate the distance OP. [3]

$$s = \int_0^{10\ln 5} 10e^{-0.1t} - 2 \, dt$$

$$= \left[\frac{10e^{-0.1t}}{-0.1} - 2t \right]_0^{10\ln 5}$$

$$= -100e^{-0.1(10\ln 5)} - 20\ln 5 + 100$$

$$= -100e^{-\ln 5} - 20\ln 5 + 100$$

$$= -20 - 20\ln 5 + 100$$

$$= 47.8\text{m}$$

(iii) Explain why the particle is again at O at some instant during the fiftieth second after first passing through O. [3]

when $t = 49$

$s = -100e^{-0.1(49)} - 2(49) + 100$
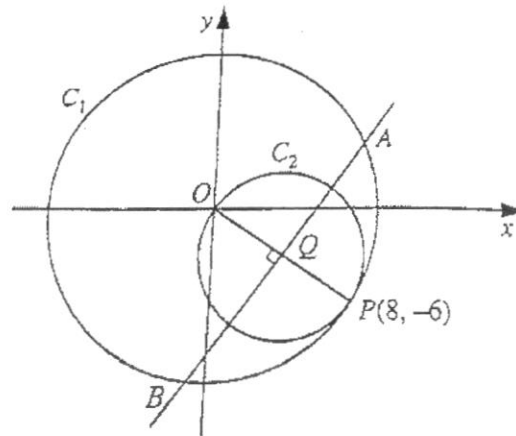
$\quad = 1.255m$

when $t - 50$

$s - -100e^{-0.1(50)} - 2(50) + 100$

$\quad = -0.674m$

Since displacement of the particle is positive when $t - 49$ and negative when $t - 50$, this shows that the particle must have passed through O at some point in the fiftieth second.

Thus the particle is again at O at some instant during the fifitieth second after passing through O.

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The diagram shows two circles C_1 and C_2.

Circle C_1 has its centre at the origin O.

Circle C_2 passes through O and has its centre at Q.

The point $P(8, 6)$ lies on both circles and OP is a diameter of C_2.

(i) Find the equation of C_1.   [2]

$$|OP| = \sqrt{(8-0)^2 + (6-0)^2}$$
$$= 10$$

Equation of $C_1 : x^2 - y^2 = 100$

(ii) Explain why the equation of C_2 is $x^2 + y^2 - 8x + 6y - 0$.   [3]

$$x^2 + y^2 - 8x + 6y - 0$$
$$x^2 - 8x + y^2 + 6y - 0$$
$$(x-4)^2 - 16 + (y+3)^2 - 9 - 0$$
$$(x-4)^2 + (y-3)^2 - 5^2$$

Centre $= (4, -3)$ because it is the mid point of OP

Radius is 5 units because it is $\dfrac{1}{2}|OP|$

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(iii) The line through Q perpendicular to OP meets the circle C_1 at the point A and

Show that the x-coordinates of A and B are $a+b\sqrt{3}$ and $a-b\sqrt{3}$ respectively, where a and b are integers to be found. [7]

$$\text{gradient } OP = -\frac{6}{8}$$

$$= -\frac{3}{4}$$

$$\text{gradient } AB = \frac{4}{3}$$

$$\frac{4}{3}(x-4) = y \div 3$$

$$4x - 16 = 3y \div 9$$

$$y = \frac{4}{3}x - \frac{25}{3} \quad \dots\dots(1)$$

$$x^2 \div y^2 = 100$$

$$y^2 = 100 - x^2 \dots(2)$$

$$(1)^2 \quad y^2 = \frac{16}{9}x^2 - \frac{200}{9}x + \frac{625}{9} \quad \dots\dots(3)$$

$$(2) = (3)$$

$$\frac{16}{9}x^2 - \frac{200}{9}x \div \frac{625}{9} = 100 - x^2$$

$$\frac{25}{9}x^2 - \frac{200}{9}x - \frac{275}{9} = 0$$

$$x^2 - 8x - 11 = 0$$

$$x = \frac{8 \pm \sqrt{64 \div 44}}{2}$$

$$= \frac{8 \div \sqrt{108}}{2}$$

$$= \frac{8 \pm 6\sqrt{3}}{2}$$

$$= 4 \div 3\sqrt{3}$$

x-coordinate of A is $4 \div 3\sqrt{3}$

x-coordinate of B is $4 - 3\sqrt{3}$

END OF PAPER

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