RIVER VALLEY HIGH SCHOOL
2020 JC2 Preliminary Examination
Higher 2

NAME	

CLASS							INDEX NUMBER		

MATHEMATICS

9758/01

Paper 1

17 September 2020

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

READ THESE INSTRUCTIONS FIRST

Write your class, index number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions.
Write your answers in the spaces provided in the question paper.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of an approved graphing calculator is expected, where appropriate.
Unsupported answers from a graphing calculator are allowed unless a question specifically states otherwise.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.
You are reminded of the need for clear presentation in your answers.
The number of marks is given in brackets [ ] at the end of each question or part question.
The total number of marks for this paper is 100.

For examiner's use only	
Question number	Mark
Total	

Calculator Model:

This document consists of X printed pages and X blank pages.

1. Let $f(x)=\ln(\cos x)$. Using the standard series, find the Maclaurin series for $f(x)$, up to and including the term in x^4. Hence, find the Maclaurin series of $\tan x$ up to and including the term in x^3. [6]

2. (a) Sketch on the same axes, the graphs of $y=1-x-\dfrac{1}{x-1}$ and $y=|ax-a|$, where $-1<a<1$. State clearly the equation of any asymptotes and coordinates of any turning points and axial intercepts. [5]

(b) Suppose that a can be any real number, state the largest possible set of solution for the inequality $1-x-\dfrac{1}{x-1}>|ax-a|$, and the corresponding range of values of a. [2]

3. The vectors **u** and **v** are unit vectors. The angle θ between **u** and **v** is an acute angle.

(i) Show that $(\mathbf{u}\cdot\mathbf{v}+|\mathbf{u}\times\mathbf{v}|)^2=1+\sin 2\theta$. [3]

(ii) The vector **w** is such that **u** + **w** = k**v**, where k is a constant in terms of θ. By considering $(\mathbf{u}+\mathbf{w})\cdot\mathbf{v}$ or otherwise, find the value of k if **w** is also a unit vector. [3]

(iii) Deduce the range of possible values of k. [1]

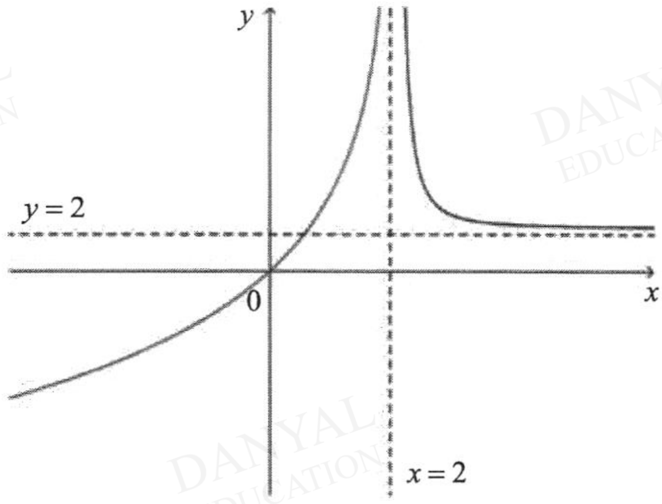
4. **(i)** State a pair of transformations that will transform the graph of $y = f(x)$ onto the graph of $y = f(2x) - 1$. [2]

It is given that

$$f(x) = \begin{cases} x^2 & , x \le 2 \\ 2 - \dfrac{1}{x-2} & , x > 2. \end{cases}$$

(ii) Sketch on separate diagrams, the graphs of $y = f(x)$ and $y = f(2x) - 1$. [4]

(iii) It is proposed that the following graph is the graph of $y = f'(x)$.



Give two reasons to explain why the proposed graph is not suitable to be the graph of $y = f'(x)$. [2]

5. **(a)** The curve C has parametric equations

$$x = \cos t, \quad y = \sec t + \sin t,$$

where $0 < t < \frac{\pi}{2}$.

Find $\int y \, dx$ in terms of x. [5]

(b) Find the exact value of $\int_{-\sqrt{2}}^{1} \left| x^3 + 1 \right| dx$. [3]

6. An *astroid* is a curve with parametric equations

$$x = a\cos^3 t,\ \ y = a\sin^3 t.$$

(i) Sketch an astroid with $a = 2$ and $0 \le t \le \dfrac{\pi}{2}$, indicating clearly the coordinates of all axial intercepts. [2]

For the rest of the question, take $a = 1$.

(ii) Find the equation of the tangent to the curve at the point P where $t = p$, simplifying your answer. [4]

(iii) The tangent to the curve at P intersects the x- and y-axes at the points Q and R respectively. Find the cartesian equation for the locus of the midpoint of QR. [3]

7. (i) Given that $\mathrm{f}(r) = \ln(3r+1)$, show that $\mathrm{f}(r) - \mathrm{f}(r-1) = \ln\left(\dfrac{3r+1}{3r-2}\right)$. [1]

(ii) Hence evaluate $\displaystyle\sum_{r=1}^{n} \ln\sqrt{\frac{3r+1}{3r-2}}$. [4]

(iii) Show that $\displaystyle\sum_{r=4}^{n-5} \ln\sqrt{\frac{3r+16}{3r+13}} = \sum_{r=9}^{n} \ln\sqrt{\frac{3r+1}{3r-2}}$. Hence find $\displaystyle\sum_{r=4}^{n-5} \ln\sqrt{\frac{3r+16}{3r+13}}$ in terms of n. [4]

8. The curves C_1 and C_2 are given by the equations $4x^2 + y^2 = 16$ and $y = x^2 - x - 2$ respectively. The region R is bounded by C_1 and C_2, and $x \ge 0$.

(a) (i) Using the substitution $x = 2\sin u$, find $\int \sqrt{16-4x^2}\ \mathrm{d}x$. [4]

(ii) Hence find the exact area of R. [4]

(b) The region S is bounded by C_2, the lines $x = -1$ and $x = 1$, and the x-axis. Find the volume generated when S is rotated 2π radians about the x-axis. [2]

9. (a) Solve the simultaneous equations

$$z - 2w^* = \mathrm{i},$$

$$\mathrm{i}z - w = \mathrm{i}.$$ [5]

(b) u and v are complex numbers such that $u = -1 - \sqrt{3}\mathrm{i}$, $|v| = \sqrt{2}$ and $\arg(v) = \dfrac{7\pi}{12}$.

(i) Find the modulus and argument of u. [2]

(ii) Express vu^* in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $-\pi < \theta \le \pi$. [2]

(iii) Find the least positive integer n such that $(vu^*)^n$ is purely imaginary. [2]

10. Sales agent A started work on 1st June 2020. He plans to acquire 2 clients in his first month of work. For subsequent months, the number of clients he plans to acquire per month is 3 more than that in the previous month.

(i) Find the number of clients agent A would acquire in June 2021. [2]

(ii) Find the number of months needed for agent A to acquire at least 500 clients in total. [3]

Sales agent B has a total 500 clients on 1st June 2020. Unlike agent A, he acquires 3 new clients every month. However, due to strong competition from other sales agents, agent B loses 1% of his clientele at the end of each month when the clients choose not to continue with his service from the following month.

(iii) Show that the clientele size of agent B at the end of n^{th} month is given by

$$297+203(0.99)^n.$$ [4]

(iv) The clientele size of agent B will eventually stabilise. State the value of this clientele size. [1]

(v) Agent B wishes to keep a clientele size of at least 350. Find the maximum percentage loss of clients he can have at the end of each month in order to meet this target. [3]

11. **(a)** Sketch the graph of $y=(\sin x+1.1)^{-0.5}$ for $0 \le x \le 2\pi$ and state the coordinates of the maximum and minimum points of the curve. [3]

(b) A comet is an icy celestial body that usually orbits the sun in the *Kuiper Belt* – a region of the solar system beyond the planet Neptune. Some comets have highly elliptical orbits that bring them into the inner solar system for short periods of time.

For a particular comet, the distance between the comet and the sun, r astronomical units (AU), at time t years after it is first observed, is described by the equation

$$r=24\left(\sin\left(\frac{t}{76}\right)+1.1\right)^{0.5}.$$

As the comet approaches the sun, it will gain kinetic energy but lose an equivalent amount of gravitational potential energy, such that the total amount of energy remains constant. This gives the equation

$$E=\frac{1}{2}mv^2-\frac{GMm}{r},$$

where E is the total energy in the system, m is the mass of the comet, G is the universal gravitational constant, M is the mass of the sun and v is the speed of the comet. The constants E, m, G and M are all positive.

(i) Show that $v^2=\dfrac{GM}{12\left(\sin\left(\frac{t}{76}\right)+1.1\right)^{0.5}}+\dfrac{2E}{m}$. [1]

(ii) By differentiation and using the result in part **(a)**, find the maximum speed of the comet in terms of the constants E, m, G and M. [5]

The luminance L of an object is a measure of how much light it is emitting. As the comet leaves the inner solar system, it reflects less light from the sun and becomes less luminous.

(iii) It is known that, when first observed, the luminance of the comet would decrease by 2 units for each astronomical unit it travelled away from the sun. Using the relationship between r and t, find the rate at which the luminance of the comet is decreasing when it is first observed. [3]

END OF PAPER

RIVER VALLEY HIGH SCHOOL
2020 JC2 Preliminary Examination
Higher 2

NAME	

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MATHEMATICS 9758/02

Paper 2 **21 September 2020**

3 hours

Candidates answer on the Question Paper
Additional Materials: List of Formulae (MF26)

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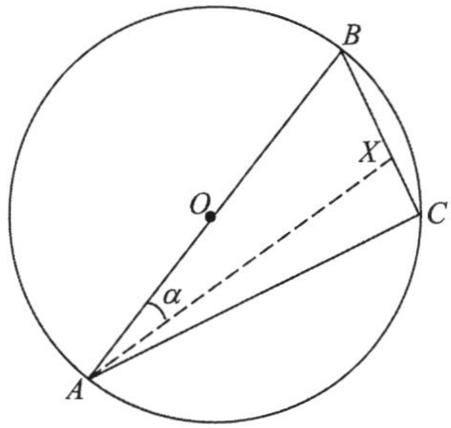
Section A: Pure Mathematics [40 marks]

1.



The diagram above shows a circle with centre O. The diameter AB has length 13 units and the point C on the circumference of the circle is such that AC is 12 units. X is a point on BC such that angle BAX is α radians.

(i) Show that $\tan \angle BAC = \dfrac{5}{12}$. [1]

(ii) By considering triangle XAC, or otherwise, show that $CX = \dfrac{60 - 144\tan\alpha}{12 + 5\tan\alpha}$. [3]

(iii) It is given that α is sufficiently small for α^3 and higher powers to be ignored, show that $CX \approx 5 + p\alpha + q\alpha^2$, where p and q are exact constants to be determined. [3]

(iv) Explain why $\alpha = 0.6$ is not suitable to be used to find CX in part **(iii)**. [1]

2. A particle moving in a liquid is such that after t seconds, its velocity is v ms^{-1}, $v \neq 0$. v satisfies the differential equation $v\dfrac{dv}{dt} + 2v^2 = te^{-2t}$.

(i) By using the substitution $y = ve^{2t}$, show that $y\dfrac{dy}{dt} = te^{2t}$. [3]

(ii) Hence find v^2 in terms of t where $\dfrac{dv}{dt} = -8$ when $t = 0$. [6]

3. A function f is said to be *involuting* or *self-inverse* if $\mathrm{f}=\mathrm{f}^{-1}$ for all x in the domain of f.

The function f is defined by

$$\mathrm{f}: x \mapsto \frac{2x+p}{x-2} \text{ for } x \in \mathbb{R}, x<2 ,$$

where p is a positive constant.

(i) Show that f is an *involuting* function. [2]

(ii) By finding $\mathrm{f}^3(x)$ and $\mathrm{f}^4(x)$, deduce $\mathrm{f}^{2021}(x)$. [3]

For the rest of the question, let $p=2$.

Another function g is defined by $\mathrm{g}: x \mapsto \ln(4-x), x<3$.

Only one of the composite functions fg and gf exists.

(iii) Give the rule of the one that exists and explain why the other does not exist. [4]

(iv) Hence find the exact range of the composite function that exists. [2]

4. The plane p has equation $3x-4y-z=-7$ and the line l has equation $\frac{x-9}{a}=\frac{y+5}{b}=z-2$, where a and b are non-zero constants. The point A with position vector $9\mathbf{i}-5\mathbf{j}+2\mathbf{k}$ lies on l.

(i) Determine a relationship between a and b if l is parallel to p. [3]

(ii) In the case where $a=2$ and $b=-2$,

(a) find the position vector of the point of intersection of l and p, [2]

(b) find the position vector of the foot of perpendicular from A to p and hence find the shortest distance from A to p. [5]

(c) find the acute angle between l and p using the results in part **(b)**. [2]

Section B: Statistics [60 marks]

5. **(a)** In how many ways can 3 letters from the word "*HAPPY*" be arranged in a row if at least one of the letters is P ? [3]

(b) A photographer is arranging 4 male and 5 female teachers for a photo shoot. Find the number of ways the photographer can arrange the teachers side by side in a straight line if

(i) one particular male teacher insists on standing in the middle, [1]

(ii) the female teachers are arranged in the order from the shortest on the left to the tallest on the right, and no two female teachers are standing next to each other, (You may assume that the heights of the female teachers are all different.) [2]

(iii) the male teachers are standing next to each other and so are the female teachers. [2]

6. A set of 36 cards is made up of cards chosen from a few packs of ordinary playing cards. The breakdown of the number of cards in each suit and denomination is given in the following table.

Denomination \ Suit	Spade	Heart	Diamond	Club
Ace	1	2	2	1
King	2	2	2	3
Queen	2	4	4	3
Jack	1	3	2	2

For example, there are 2 Aces of Diamond and 3 Jacks of Heart in the set of 36 cards.

(i) This set of 36 cards is put into a bag and 1 card is selected at random.

(a) Find the probability that the card is either an Ace or a Jack but not a Club. [1]

(b) Find the probability that the card is neither a Jack nor a Heart. [1]

(ii) The cards are all placed back into the bag and 2 cards are selected at random.

(a) Find the probability that both cards are Aces given that neither of the cards is Spade. [2]

(b) Find the probability that the 2 cards include exactly one Queen and exactly one Diamond. [3]

7. An infectious virus called Zeevus hit an Asian country Atopia. It was found from data corroborated with other countries in Asia that the virus had a *mean incubation period* of 10.8 days.

The incubation period is defined as the time from exposure to a virus to the development of symptoms. For highly infectious diseases, authorities are guided by the mean incubation period to quarantine suspected infected patients in order to minimise transmission.

A city of Atopia reported that the incubation period had been found to be as long as 20 days in a few cases and this had worried the authorities. A medical researcher was tasked to collect a random sample of 100 patients. The incubation period, x days, are summarised as follows:

$$\sum(x-12)=39,\ \sum(x-12)^2=2716.88.$$

(i) Calculate the unbiased estimates of the population mean and variance of the incubation period. [2]

(ii) Explain whether the researcher should carry out a 1-tail or 2-tail test, stating the hypotheses for the tests, and defining any symbols you use. [2]

(iii) Carry out the test, at 1% level of significance for the researcher. [3]

A few months later, the authorities suspected that a new strain of the same virus might have developed which affected the incubation period. They assigned the same researcher to find out if there was a change in the mean incubation period. Data from 100 randomly selected patients from across the country were collected for the test.

(iv) Determine the set of values of the sample mean to show that the incubation period of the new strain of the virus is different from the original strain at the 1% level of significance. [3]

8. A company has 50 workers. It is known that, on average, the probability that any particular worker is on sick leave on a given day is p. Let W be the random variable of the number of workers on sick leave on a given day.

(i) State, in the context of the question, two assumptions needed to model W by a binomial distribution. Explain why one of the assumptions may not be valid in this context. [3]

Assume that the two assumptions in part **(i)** do in fact hold. A day is selected at random and the probability that at most 2 workers are on sick leave is 0.54.

(ii) Write down an equation satisfied by p. Hence find the value of p. [3]

Assuming there are 60 working days in 2 months,

(iii) find the expected number of days in 2 months for which at most 2 workers are on sick leave. [2]

(iv) find the probability that the mean number of workers on sick leave per day over 2 months is at least 2. [3]

9. A local radio station organises a call-in competition, in which listeners are given the names of 7 local celebrities and they are told that 4 of them are in the studio. Listeners will then call in and give the names of the 4 celebrities they think are in the studio. The random variable X is the number of correct guesses made by a randomly selected listener.

(i) Show that $\mathrm{P}(X=1)=\dfrac{4}{35}$. [1]

(ii) Find the probability distribution of X. [3]

(iii) Find $\mathrm{E}(X)$ and show that $\mathrm{Var}(X)=\dfrac{24}{49}$. [2]

On any day, 3 listeners will be selected to participate in this competition. The radio station will only reveal which 4 celebrities are in the studio after all 3 listeners have made their guesses.

(iv) Find the probability that the total number of correct guesses from the 3 listeners is 6. State any assumption you have made in your calculation. [4]

(v) Find the probability that all 3 listeners make the same guesses. [2]

10. **In this question you should state the parameters of any distributions that you use.**

A brand of mobile phone launches a new model. Each phone is made up of two key parts – phone body and removable battery. The masses in grams of the phone bodies have the distribution $N(165, 15^2)$.

(i) Sketch the distribution for masses of the phone bodies within three standard deviations of the mean. [2]

(ii) Find the probability that the difference between 2 randomly chosen phone bodies is less than 0.5 grams. [3]

It is given further that the mass of a battery is modelled as 18% of the mass of a phone body.

(iii) According to ergonomics standard, the ideal mass of a phone, **with the battery installed**, is between 140 and 170 grams. Find the probability that a randomly chosen phone is of ideal mass. [3]

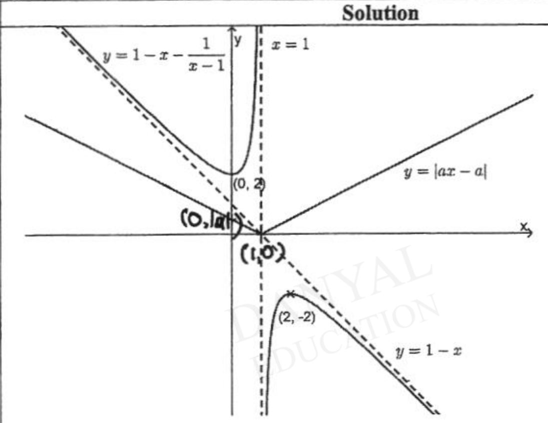
A standard travel charger is also enclosed with the phone and it weighs q grams.

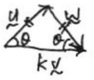
(iv) The probability that the total mass of a phone with a standard travel charger is less than 0.247 kg is 0.85. Find the mass of the travel charger. [4]

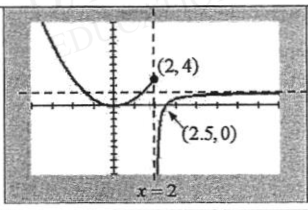
END OF PAPER

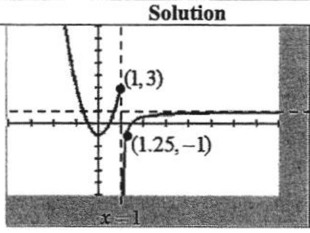
2020 RVHS H2 Math Prelim P1 Solution

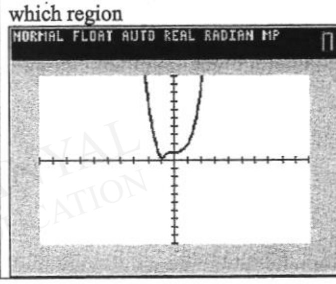
Q1	Solution	Mark Scheme	Comments
	$\mathrm{f}(x)=\ln(\cos x)$ $=\ln\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots\right)$ $=\left(-\frac{x^2}{2!}+\frac{x^4}{4!}\right)-\frac{1}{2}\left(-\frac{x^2}{2!}+\frac{x^4}{4!}\right)^2+\dots$ $=-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{1}{2}\left(\frac{x^4}{4}+2\left(-\frac{x^2}{2!}\right)\left(\frac{x^4}{4!}\right)+\left(\frac{x^4}{4!}\right)^2\right)+\dots$ $\approx-\frac{x^2}{2}-\frac{x^4}{12}$	M1: $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\dots$ M1: applying $\ln(1+x)=x-\frac{1}{2}x^2+\dots$ A1	Applying standard series question. Obvious as to which series to apply.
	Since $\frac{\mathrm{d}}{\mathrm{d}x}\ln(\cos x)=\frac{-\sin x}{\cos x}=-\tan x$, $\tan x=-\frac{\mathrm{d}}{\mathrm{d}x}\ln(\cos x)$ $=-\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{x^2}{2}-\frac{x^4}{12}+\right)$ $=-\left(-x-\frac{x^3}{3}\right)+$ $\approx x+\frac{x^3}{3}$	B1: $\frac{\mathrm{d}}{\mathrm{d}x}\ln(\cos x)=-\tan x$ M1: differentiate part (i) ans wrt x A1	One possible clue to spot differentiate is first part is up to x^4, while the 2nd part ask for up to x^3.

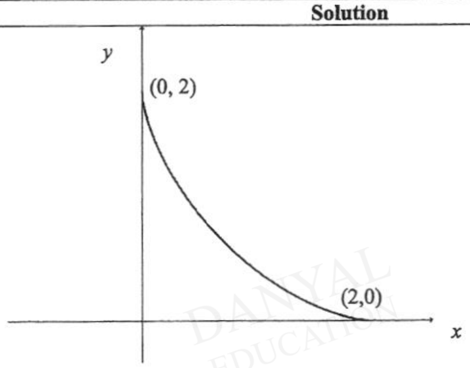
Q2	Solution	Mark Scheme	Comments
(a)		B1: equations of asymptotes B1: shape of graph $y=1-x-\frac{1}{x-1}$ B1: turning points (0, 2) and (2, –2) B1: shape of graph $y=\lvert ax-a\rvert$, kink at positive x axis B1: axial intercepts (1, 0) and $(0,\lvert a\rvert)$	Standard question. For modulus graph, do use the GC and sub in a as a fixed value to get a sense of the shape.
(b)	$y=\lvert ax-a\rvert$ is always more than $y=1-x-\frac{1}{x-1}$ for $x>1$ based on the sketch. For $x<1$, as long as the gradient of $y=\lvert ax-a\rvert$ is less than that of the asymptote $y=1-x$, then $y=1-x-\frac{1}{x-1}$ will be more than $y=\lvert ax-a\rvert$ for all the x-values in the range $x<1$. Maximal solution set = $\{x\in\ \ :x<1\}$, for $-1\le a\le 1$.	B1 B1	A question that requires the interpretation of the unknown a and how it affects the intersection.

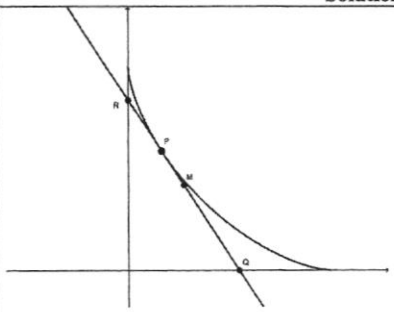
Q3	Solution	Mark Scheme	Comments
(i)	$(\mathbf{u}\cdot\mathbf{v}+\lvert\mathbf{u}\times\mathbf{v}\rvert)^2$ $=(\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\cos\theta+\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\sin\theta)^2$ $=(\sin\theta+\cos\theta)^2$ $=\cos^2\theta+2\cos\theta\sin\theta+\sin^2\theta$ $=1+\sin 2\theta$	B1: $\mathbf{u}\cdot\mathbf{v}=\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\cos\theta$ or $\lvert\mathbf{u}\times\mathbf{v}\rvert=\lvert\mathbf{u}\rvert\lvert\mathbf{v}\rvert\sin\theta$ B1: recognising that **u** and **v** are unit vectors and simplify to an expression without **u** and **v** B1: expand and simplify using double angle	Tricky question that requires us to apply definition and unit vector before expansion. Note that there is no way to simplify $(\mathbf{u}\cdot\mathbf{v})^2$ or $\lvert\mathbf{u}\times\mathbf{v}\rvert^2$ except to use definition. So we just have to use definition.
(ii)	$(\mathbf{u}+\mathbf{w})\cdot\mathbf{v}=k\mathbf{v}\cdot\mathbf{v}=k\lvert\mathbf{v}\rvert^2$ $(\mathbf{u}+\mathbf{w})\cdot\mathbf{v}=k$ $\mathbf{u}\cdot\mathbf{v}+\mathbf{w}\cdot\mathbf{v}=k$ $\cos\theta+\cos\theta=k$ $k=2\cos\theta$ 	B1: sub $\mathbf{u}+\mathbf{w}=k\mathbf{v}$ and get $k\lvert\mathbf{v}\rvert^2$ B1: $\mathbf{u}\cdot\mathbf{v}=\cos\theta$ or $\mathbf{w}\cdot\mathbf{v}=\cos\theta$ B1: $k=2\cos\theta$	Follow the hint given in the question and apply dot product expansion and unit vector
(iii)	Since $0<\theta<90°$, $0<k<2$	B1: $0<k<2$	

Q4	Solution	Mark Scheme	Comments
(i)	Scale parallel to x-axis by a factor of $\frac{1}{2}$. Translate 1 unit in the negative y-direction.	B1 B1 *(OR in reverse sequence.)*	
(ii)		$y=\mathrm{f}(x)$ B1: correct graph for $x<2$ B1: correct graph for $x>2$ –1 for any missing labelling	

Q4	Solution	Mark Scheme	Comments
		$y=\mathrm{f}(2x)-1$ B1: correct graph for $x<2$ B1: correct graph for $x>2$ –1 for any missing labelling	
(iii)	For the graph of $y=\mathrm{f}'(x)$, 1. there should be a horizontal asymptote $y=0$ instead of $y=2$, **OR** 2. it should end at (2, 4) for the segment for which $x\le 2$ since $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right\rvert_{x=2}=2(2)=4$, **OR** 3. the segment for which $x\le 2$ should be a straight line (with equation $y=2x$). (Any 2 reasons)	B1 B1 Any 2 reasons	

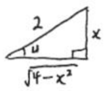
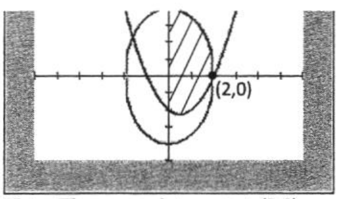
Q5	Solution	Mark Scheme	Comments
(a)	$\int y \, \mathrm{d}x = \int (\sec t + \sin t)(-\sin t)\, \mathrm{d}t$ $= \int -\tan t - \sin^2 t \, \mathrm{d}t$ $= -\int \tan t \, \mathrm{d}t - \int \sin^2 t \, \mathrm{d}t$ $= -\int \tan t \, \mathrm{d}t - \int \frac{1-\cos 2t}{2} \, \mathrm{d}t$ $= -\ln\lvert\sec t\rvert - \frac{t}{2} + \frac{1}{4}\sin 2t + c$ $= \ln\lvert\cos t\rvert - \frac{t}{2} + \frac{1}{2}\sin t \cos t + c$ $= \ln\lvert x\rvert - \frac{1}{2}\cos^{-1} x + \frac{x}{2}\sqrt{1-x^2} + c$	B1: $\int y \, \mathrm{d}x = \int (\sec t + \sin t)(-\sin t)\, \mathrm{d}t$ B1: $\int \tan t \, \mathrm{d}t = \ln\lvert\sec t\rvert + c$ B1: $\int \sin^2 t \, \mathrm{d}t = \int \frac{(1-\cos(2t))}{2} \, \mathrm{d}t$ M1: sub $\sin t$ back to x A1	
(b)	$\int_{-\sqrt{2}}^{1} \lvert x^3 + 1\rvert \, \mathrm{d}x$ $= \int_{-\sqrt{2}}^{-1} -(x^3+1)\, \mathrm{d}x + \int_{-1}^{1} x^3 + 1 \, \mathrm{d}x$ $= -\left[\frac{x^4}{4} + x\right]_{-\sqrt{2}}^{-1} + \left[\frac{x^4}{4} + x\right]_{-1}^{1}$ $= -\left(\frac{1}{4} - 1 - 1 + \sqrt{2}\right) + \left(\frac{1}{4} + 1 - \frac{1}{4} + 1\right)$ $= \frac{15}{4} - \sqrt{2}$	M1: split into 2 regions with correct limits M1: integration and substitute limits A1: exact	Remember to key in GC to check which region 

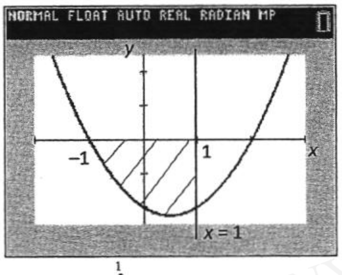
Q6	Solution	Mark Scheme	Comments
(i)	 	B1: shape of curve B1: axial intercepts (0, 2) and (2, 0)	
(ii)	$\frac{\mathrm{d}x}{\mathrm{d}t} = 3\cos^2 t(-\sin t)$ $\frac{\mathrm{d}y}{\mathrm{d}t} = 3\sin^2 t(\cos t)$ $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3\sin^2 t \cos t}{-3\sin t \cos^2 t} = -\tan t$ P is the point $(\cos^3 p, \sin^3 p)$ Thus, the equation of the tangent: $y - \sin^3 p = -\tan p(x - \cos^3 p)$ $y = (-\tan p)x + \sin p \cos^2 p + \sin^3 p$ $y = (-\tan p)x + \sin p(\cos^2 p + \sin^2 p)$ $y = (-\tan p)x + \sin p$	M1: finding $\frac{\mathrm{d}y}{\mathrm{d}x}$ using $\frac{\mathrm{d}y}{\mathrm{d}t}$ and $\frac{\mathrm{d}}{t}$ M1: finding value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at $t = p$ and coordinates of P M1: applying equation of tangent correctly A1: simplified	

Q6	Solution	Mark Scheme	Comments
(iii)			
	$Q(\cos p, 0)$, $R(0, \sin p)$ Let the midpoint of QR be M. Then $M\left(\frac{1}{2}\cos p, \frac{1}{2}\sin p\right)$	M1: mid-point formula	
	Therefore, parametric equations for the locus of M: $x = \frac{1}{2}\cos p,\ \ y = \frac{1}{2}\sin p$	M1: express x and y in terms of p	
	Therefore, Cartesian equation for the locus of M: $\cos^2 p + \sin^2 p = 1$ $(2x)^2 + (2y)^2 = 1$ $x^2 + y^2 = \left(\frac{1}{2}\right)^2$	A1: Cartesian Equation	

Q7	Solution	Mark Scheme	Comments
(i)	$f(r) - f(r-1)$ $= \ln(3r+1) - \ln(3(r-1)+1)$ $= \ln(3r+1) - \ln(3r-2)$ $= \ln\left(\frac{3r+1}{3r-2}\right)$ (shown)	B1: show clearly	
(ii)	$\sum_{r=1}^{n} \ln\sqrt{\frac{3r+1}{3r-2}}$ $= \frac{1}{2}\sum_{r=1}^{n} \ln\left(\frac{3r+1}{3r-2}\right)$	B1: $\frac{1}{2}\sum_{r=1}^{n} \ln\left(\frac{3r+1}{3r-2}\right)$	
	$= \frac{1}{2}\sum_{r=1}^{n}\left[f(r) - f(r-1)\right]$ $= \frac{1}{2}\left[\begin{array}{c} \cancel{f(1)} - f(0) \\ \cancel{f(2)} - \cancel{f(1)} \\ \cancel{f(3)} - \cancel{f(2)} \\ \\ \cancel{f(n-1)} - \cancel{f(n-2)} \\ f(n) - \cancel{f(n-1)} \end{array}\right]$ $= \frac{1}{2}\left[f(n) - f(0)\right]$	M1: MOD	
	$= \frac{1}{2}\left[\ln(3n+1) - \ln(3(0)+1)\right]$	M1: in terms of n and constant	
	$= \frac{1}{2}\ln(3n+1)$	A1: simplified	

Q7	Solution	Mark Scheme	Comments
(iii)	$\displaystyle\sum_{r=4}^{n-5}\ln\sqrt{\frac{3r+16}{3r+13}}=\sum_{r-5=4}^{n-5}\ln\sqrt{\frac{3(r-5)+16}{3(r-5)+13}}$ replace r by $r-5$ $\displaystyle=\sum_{r=9}^{n}\ln\sqrt{\frac{3r+1}{3r-2}}$ (shown) $\displaystyle\therefore\sum_{r=4}^{n-5}\ln\sqrt{\frac{3r+16}{3r+13}}=\sum_{r=1}^{n}\ln\sqrt{\frac{3r+1}{3r-2}}-\sum_{r=1}^{8}\ln\sqrt{\frac{3r+1}{3r-2}}$ $\displaystyle=\frac{1}{2}\ln(3n+1)-\frac{1}{2}\ln(3(8)+1)$ $\displaystyle=\frac{1}{2}\ln\left(\frac{3n+1}{25}\right)$	B1: change of variables (r by $r-5$) B1: $\displaystyle\sum_{r=1}^{n}\ln\sqrt{\frac{3r+1}{3r-2}}-\sum_{r=1}^{8}\ln\sqrt{\frac{3r+1}{3r-2}}$ M1: applying part (i) A1	

Q8	Solution	Mark Scheme	Comments
(a)(i)	Let $x=2\sin u$. Then $\frac{dx}{du}=2\cos u$. Thus, $\int\sqrt{16-4x^2}\ dx=\int\sqrt{16-4(2^2\sin^2 u)}\,(2\cos u)\ du$ $=8\int\sqrt{1-\sin^2 u}\,(\cos u)\ du$ $=8\int\cos^2 u\ du$ $=4\int\cos 2u+1\ du$ $=4\left[\frac{1}{2}\sin 2u+u\right]+c$ $=4\sin u\cos u+4u+c$ $=2x\frac{\sqrt{4-x^2}}{2}+4\sin^{-1}\left(\frac{x}{2}\right)+c$  $=x\sqrt{4-x^2}+4\sin^{-1}\left(\frac{x}{2}\right)+c$	B1: $\int\sqrt{16-4(2^2\sin^2 u)}\,(2\cos u)\ du$ M1: double angle formula to integrate M1: sub $\cos u$ back to x A1	
(a)(ii)	 Note: The curves intersect at (2,0).	B1: curves intersect at (2, 0)	

Q8	Solution	Mark Scheme	Comments
	$\text{Area} = \int_0^2 \sqrt{16-4x^2} - \left(x^2 - x - 2\right)\, \mathrm{d}x$ $= \left[x\sqrt{4-x^2} + 4\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{3}x^3 + \frac{1}{2}x^2 + 2x \right]_0^2$ $= \left[2\pi - \frac{8}{3} + 2 + 4 \right]$ $= 2\pi + \frac{10}{3}$ units^2	M1: formulate as a difference of the 2 curves expressions M1: use of part (i) A1	
(b)	 $\text{Volume} = \pi \int_{-1}^{1} y^2\, \mathrm{d}x$ $= \pi \int_{-1}^{1} \left(x^2 - x - 2\right)^2\, \mathrm{d}x$ $= 20.1$ units^3 (3 s.f.)	B1: $\pi \int_{-1}^{1} \left(x^2 - x - 2\right)^2\, \mathrm{d}x$ B1: 20.1	Remember to use GC to get the answer

Q9	Solution	Mark Scheme	Comments
(a)	$z - 2w^* = \mathrm{i}\quad (1)$ $\mathrm{i}z - w = \mathrm{i}\quad (2)$ From (2), $\mathrm{i}z = w + \mathrm{i}$ $z = 1 - w\mathrm{i}\quad (3)$ sub (3) into (1): $1 - w\mathrm{i} - 2w^* = \mathrm{i}$ let $w = a + b\mathrm{i}$, $1 - (a + b\mathrm{i})\mathrm{i} - 2(a - b\mathrm{i}) = \mathrm{i}$ $1 - a\mathrm{i} + b - 2a + 2b\mathrm{i} = \mathrm{i}$ $1 - 2a + b + (2b - a)\mathrm{i} = \mathrm{i}$ Comparing real part, $1 - 2a + b = 0$ $2a - b = 1\quad (4)$ Comparing imaginary part, $2b - a = 1\quad (5)$ Solving (4) and (5) using GC, $a = 1, b = 1$ Therefore, $w = 1 + \mathrm{i}$, $z = 1 - (1 + \mathrm{i})\mathrm{i}$ $= 2 - \mathrm{i}$	M1: substitution or elimination to make equation of 1 variable M1: let z or w be $a + bi$ M1: compare real and imaginary parts A1: $w = 1 + \mathrm{i}$ A1: $z = 2 - \mathrm{i}$	
(b)(i)	$\lvert u \rvert = \sqrt{(-1)^2 + \left(-\sqrt{3}\right)^2} = 2$ $\arg(u) = -\pi + \tan^{-1}\left(\frac{\sqrt{3}}{1}\right)$ $= -\frac{2\pi}{3}$	B1 B1	

<table>
<tr><th>Q9</th><th>Solution</th><th>Mark Scheme</th><th>Comments</th></tr>
<tr><td>(b)(ii)</td><td>$|vu^*| = |v||u^*|$
$= 2\sqrt{2}$
$\arg(vu^*) = \arg(v) + \arg(u^*)$
$= \arg(v) - \arg(u)$
$= \frac{7\pi}{12} - \left(-\frac{2\pi}{3}\right)$
$= \frac{15\pi}{12}$
$= -\frac{3\pi}{4}$
$\therefore vu^* = 2\sqrt{2}e^{i\left(-\frac{3\pi}{4}\right)}$</td><td>B1: either finding mod or arg correctly

A1</td><td></td></tr>
<tr><td>(b)(iii)</td><td>For $(vu^*)^n$ to be purely imaginary,
$\arg\left((vu^*)^n\right) = (2k+1)\frac{\pi}{2}$ for $k \in$
$n\arg(vu^*) = (2k+1)\frac{\pi}{2}$
$-\frac{3n\pi}{4} = \frac{(2k+1)\pi}{2}$
$-3n = 4k+2$
$n = \frac{-2}{3}(2k+1)$
$\therefore$ smallest positive integer $n = 2$</td><td>M1: arg $(vu^*)^n$ is $(2k+1)\frac{\pi}{2}$ for $k \in \mathbb{Z}$

A1</td><td></td></tr>
</table>

<table>
<tr><th>Q10</th><th>Solution</th><th>Mark Scheme</th><th>Comments</th></tr>
<tr><td>(i)</td><td>June 2021 is the 13th month.
Required no. of clients
$= T_3$
$= 2 + (13-1)(3)$
$= 38$</td><td>M1: applying AP term formula
A1</td><td></td></tr>
<tr><td>(ii)</td><td>Want $S_n \geq 500$
$\frac{n}{2}[2(2)+(n-1)(3)] \geq 500$
$n(3n+1) \geq 1000$
$3n^2 + n - 1000 \geq 0$
$n \leq -18.4248$ or $n \geq 18.0915$

Hence the agent needs <u>19 months</u> to achieve a clientele of at least 500.</td><td>M1: applying AP sum formula

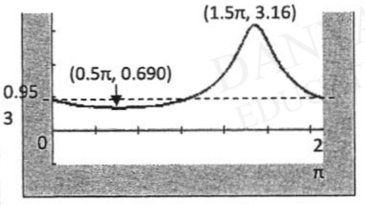
M1: solve quadratic inequality correctly
A1</td><td></td></tr>
<tr><td>(iii)</td><td><table>
<tr><th>Month</th><th>Clientele size before attrition</th><th>After attrition</th></tr>
<tr><td>1</td><td>$500+3$</td><td>$0.99(500+3)$</td></tr>
<tr><td>2</td><td>$0.99(500)+0.99(3)+3$</td><td>$(0.99)^2(500)+(0.99)^2(3)+(0.99)3$</td></tr>
<tr><td>3</td><td></td><td>$(0.99)^3(500)+(0.99)^3(3)+(0.99)^2(3)+(0.99)3$</td></tr>
<tr><td>n</td><td></td><td>$(0.99)^n(500)+(0.99)^n(3)+(0.99)^{n-1}(3)+\ldots+(0.99)3$</td></tr>
</table>
Size of clientele at end of n^{th} month
$= (0.99)^n(500)+(0.99)^n(3)+(0.99)^{n-1}(3)+\ldots+(0.99)3$
$= 500(0.99)^n + \frac{3(0.99)\left(1-0.99^n\right)}{1-0.99}$
$= 500(0.99)^n + 297\left(1-0.99^n\right)$
$= 297 + 203(0.99)^n$ (shown)</td><td>B1: list 1 term correctly after $n = 1$

M1: recognising or listing a GP pattern

M1: applying sum of GP

B1: clear steps leading to show form</td><td></td></tr>
</table>

Q10	Solution	Mark Scheme	Comments
(iv)	As $n \to \infty$, $203(0.99)^n \to 0$. So the clientele size converges to <u>297</u>.	B1: 297	
(v)	let percentage loss be $(1-x)\times 100\%$ Find x s.t. $\lim_{n\to\infty}\left[x^n(500)+x^n(3)+x^{n-1}(3)+\ldots+x(3)\right]\geq 350$ $\Rightarrow \lim_{n\to\infty}\left[(500)x^n+\frac{3x\left(1-x^n\right)}{1-x}\right]\geq 350$ $\Rightarrow \frac{3x}{1-x}\geq 350$ since $x^n \to 0$ as $n \to \infty$ $\Rightarrow x \geq 0.9915014$ Hence, the maximum percentage loss that agent B can have is $(1-0.9915014)\times 100\% = 0.84985\% = 0.850\%$ (3 sf)	M1: Set up inequality correctly M1: applying limit correctly or GP sum to infinity formula A1	

Q11	Solution	Mark Scheme	Comments
(a)	$y=(\sin x+1.1)^{-0.5}$ 	B1: shape of curve B1: $(1.5\pi, 3.16)$ and $(0.5\pi, 0.690)$ B1: start and end point is same y-value	

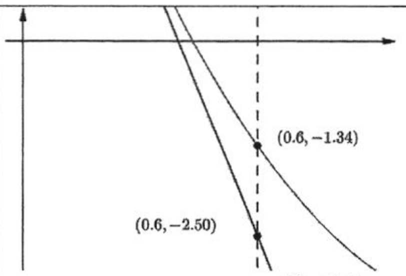
Q11	Solution	Mark Scheme	Comments
(b)(i)	$E=\frac{1}{2}mv^2-\frac{GMm}{r}$ $mv^2=2E+\frac{2GMm}{r}$ $v^2=\frac{2E}{m}+\frac{2GM}{r}=\frac{GM}{12\left(\sin\left(\frac{t}{76}\right)+1.1\right)^{0.5}}+\frac{2E}{m}$ (shown)	B1: clear steps of substitution and making v^2 the subject, leading to show form	
(b)(ii)	$v^2=\frac{GM}{12\left(\sin\left(\frac{t}{76}\right)+1.1\right)^{0.5}}+\frac{2E}{m}$ $=\frac{GM}{12}\left(\sin\left(\frac{t}{76}\right)+1.1\right)^{-0.5}+\frac{2E}{m}$ $2v\left(\frac{\mathrm{d}v}{\mathrm{d}t}\right)=-\frac{GM}{24}\left(\sin\left(\frac{t}{76}\right)+1.1\right)^{-1.5}\cdot\left(\cos\left(\frac{t}{76}\right)\right)\cdot\left(\frac{1}{76}\right)$ For stationary values, $0=\frac{-GM}{1824}\left(\sin\left(\frac{t}{76}\right)+1.1\right)^{-1.5}\cdot\left(\cos\left(\frac{t}{76}\right)\right)$ $\cos\frac{t}{76}=0$ $t=38\pi,\ 114\pi$ We don't need to find other values of t since it will have the same repeated set of speeds each time. Deducing from part (a), maximum when $t=114\pi$	B1: differentiate correctly with chain rule B1: sub $\frac{\mathrm{d}v}{\mathrm{d}t}=0$ and solve for t B1: deduce that max when $t=114\pi$	There is no need to show max because we can deduced from the graph in (a). If the question did not specify that we have to do differentiation, we can immediately deduce the t values by multiplying 76 to 0.5π and 1.5π. Because x has been replaced by $\frac{t}{76}$. Nevertheless this serves as a checking mechanism.

Q11	Solution	Mark Scheme	Comments
	When $t = 114\pi$, $v^2 = \dfrac{GM}{12\left(\sin\left(\dfrac{114\pi}{76}\right)+1.1\right)^{0.5}} + \dfrac{2E}{m}$ $= \dfrac{GM}{12\left(\sin(1.5\pi)+1.1\right)^{0.5}} + \dfrac{2E}{m}$ $v^2 = 0.263523GM + \dfrac{2E}{m}$ $v = \sqrt{0.264GM + \dfrac{2E}{m}}$	M1: sub $t = 114\pi$ to find v A1	
(b)(iii)	$\dfrac{dL}{dt} = \dfrac{dL}{dr} \cdot \dfrac{dr}{dt}$ $r = 24\left(\sin\left(\dfrac{t}{76}\right)+1.1\right)^{0.5}$ $\dfrac{dr}{dt} = 12\left(\sin\left(\dfrac{t}{76}\right)+1.1\right)^{-0.5}\left(\cos\left(\dfrac{t}{76}\right)\right)\left(\dfrac{1}{76}\right)$ $= \dfrac{3\cos\left(\dfrac{t}{76}\right)}{19\left(\sin\left(\dfrac{t}{76}\right)+1.1\right)^{0.5}}$ $\left.\dfrac{dr}{dt}\right\|_{t=0} = \dfrac{3\cos(0)}{19\left(\sin(0)+1.1\right)^{0.5}} = 0.150547$ Given $\left.\dfrac{dL}{dr}\right\|_{t=0} = -2$	B1: finding $\dfrac{dr}{dt}$ by differentiating wrt t correctly	

Q11	Solution	Mark Scheme	Comments
	$\dfrac{dL}{dt} = (-2)(0.150547) = -0.301094$ Therefore, the luminance is decreasing at a rate of 0.301 units per year at the time of discovery.	M1: applying chain rule with values substituted A1: decreasing at a rate of 0.301	

2020 RVHS H2 Math Prelim P2 Solution

Q1	Solution	Mark Scheme	Comments
(i)	ΔACB is a right-angled triangle. $\therefore BC=\sqrt{13^2-12^2}=5$ Hence $\tan\angle BAC=\dfrac{BC}{AC}=\dfrac{5}{12}$ (shown)	B1: use of Pythagoras and state clearly the right-angled triangle	
(ii)	$CX=12\tan\left(\angle BAC-\alpha\right)$ $=\dfrac{12(\tan\angle BAC-\tan\alpha)}{1+\tan\angle BAC\tan\alpha}$ $=\dfrac{12\left(\frac{5}{12}-\tan\alpha\right)}{1+\frac{5}{12}\tan\alpha}$ $=\dfrac{5-12\tan\alpha}{1+\frac{5}{12}\tan\alpha}$ $=\dfrac{60-144\tan\alpha}{12+5\tan\alpha}$ (shown)	B1: $CX=12\tan\left(\angle BAC-\alpha\right)$ B1: apply addition formula for tan B1: show clear steps to reach show	

(iii)	For small α, $\tan\alpha\approx\alpha$ $CX=\dfrac{60-144\tan\alpha}{12+5\tan\alpha}$ $\approx(60-144\alpha)(12+5\alpha)^{-1}$ $=(5-12\alpha)\left(1+\frac{5}{12}\alpha\right)^{-1}$ $=(5-12\alpha)\left(1-\frac{5}{12}\alpha+\left(\frac{5}{12}\alpha\right)^2+\ldots\right)$ $\approx 5-12\alpha-\frac{25}{12}\alpha+5\alpha^2+\frac{125}{144}\alpha^2$ $=5-\frac{169}{12}\alpha+\frac{845}{144}\alpha^2$ $p=-\frac{169}{12}$, $q=\frac{845}{144}$ (shown)	B1: $\tan\alpha\approx\alpha$ B1: $(60-144\alpha)(12+5\alpha)^{-1}$ A1	
(iv)	 By sketching the graphs of $y=\dfrac{60-144\tan\alpha}{12+5\tan\alpha}$ (red) and $y=5-\frac{169}{12}\alpha+\frac{845}{144}\alpha^2$ (blue), we observe that at $\alpha=0.6$, the two graphs departed from each other significantly. Thus, the approximation is not very good.	B1: by graph and explain	

Q2	Solution	Mark Scheme	Comments
(i)	$y = ve^{2t}$ $\frac{dy}{dt} = e^{2t}\frac{dv}{dt} + 2ve^{2t}$ (*) $v\frac{dv}{dt} + 2v^2 = te^{-2t}$ $ve^{2t}\frac{dv}{dt} + 2v^2e^{2t} = t$ $e^{2t}\frac{dv}{dt} + 2ve^{2t} = tv^{-1}$ Replace LHS with (*) $\frac{dy}{dt} = tv^{-1} = t.\frac{e^{2t}}{y}$ $y\frac{dy}{dt} = t\ e^{2t}$ (shown)	B1: $\frac{dy}{dt} = e^{2t}\frac{dv}{dt} + 2ve^{2t}$ M1: replacement B1: show steps clearly to reach show	
(ii)	$\int y\ dy = \int te^{2t}\ dt$ $\frac{y^2}{2} = \frac{te^{2t}}{2} - \int\frac{e^{2t}}{2}\ dt$ $\frac{y^2}{2} = \frac{te^{2t}}{2} - \frac{e^{2t}}{4} + c$ $y^2 = te^{2t} - \frac{e^{2t}}{2} + D$, where $2c = D$ $v^2e^{4t} = te^{2t} - \frac{e^{2t}}{2} + D$ From $v\frac{dv}{dt} + 2v^2 = te^{-2t}, t = 0, \frac{dv}{dt} = -8$,	B1: variables separable B1: correct integration B1: sub $y = ve^{2t}$ M1: sub $t = 0, \frac{dv}{dt} = -8$	

Q2	Solution	Mark Scheme	Comments
	$-8v + 2v^2 = 0$ $-8 + 2v = 0$ since $v \neq 0$ $v = 4$ Substituting the values of v and t, $v^2e^{4t} = te^{2t} - \frac{e^{2t}}{2} + D$ $16 = -\frac{1}{2} + D$ $D = 16.5$ Therefore, $v^2e^{4t} = te^{2t} - \frac{e^{2t}}{2} + 16.5$ $v^2 = te^{-2t} - \frac{e^{-2t}}{2} + (16.5)e^{-4t}$	B1: sub v found and $t = 0$ A1	

Q3	Solution	Mark Scheme	Comments
(i)	Let $y = \frac{2x+p}{x-2}, x \in \quad , x < 2,$ $y(x-2) = 2x + p$ $yx - 2x = 2y + p$ $x(y-2) = 2y + p$ $x = \frac{2y+p}{y-2}$ $\therefore\ f^{-1}(x) = \frac{2x+p}{x-2},\ x < 2$ Since $f = f^{-1}$, f is an involuting function. (SHOWN)	B1: making x the subject and show that the rule is the same B1: show that the domain is the same	

Q3	Solution	Mark Scheme	Comments
(ii)	From (i), $\mathrm{f}(x)=\mathrm{f}^{-1}(x)$ Composing function on both sides, $\mathrm{ff}(x)=\mathrm{ff}^{-1}(x)$ $\mathrm{f}^2(x)=x$ --------(*) $\mathrm{f}^3(x)=\mathrm{f}(x)=\dfrac{2x+p}{x-2}$ Rewriting (*), $\mathrm{f}^3(x)=\mathrm{f}^{-1}(x)$ since f involuting $\mathrm{f}^4(x)=\mathrm{f}^2(x)=x$ Hence, $\mathrm{f}^{2021}(x)=\mathrm{f}(x)=\dfrac{2x+p}{x-2}$	B1: $\mathrm{f}^2(x)=x$ B1: $\mathrm{f}^3(x)=\mathrm{f}(x)$ A1	
(iii)	Now $\mathrm{f}(x)=\dfrac{2x+2}{x-2}, x<2$ $\mathrm{g}(x)=\ln(4-x), x<3$ As $\mathrm{R_g}=(0,\infty)\not\subset(-\infty,2)=\mathrm{D_f}$, fg does not exist. Since $\mathrm{R_f}=(-\infty,2)\subseteq(-\infty,3)=\mathrm{D_g}$, gf exists. $\mathrm{gf}(x)=\mathrm{g}\left(\dfrac{2x+2}{x-2}\right)$ $=\ln\left(4-\dfrac{2x+2}{x-2}\right)$ $=\ln\left(\dfrac{2x-10}{x-2}\right)$ $\mathrm{gf}: x\mapsto\ln\left(\dfrac{2x-10}{x-2}\right), x\in\mathbb{R}, x<2$	B1: $\mathrm{R_g}=(0,\infty)$, $\mathrm{D_f}=(-\infty,2)$ B1: explain using subset rule correctly M1: sub f(x) into g(x) A1: gf(x)	

Q3	Solution	Mark Scheme	Comments
(iv)	Using mapping method, $\mathrm{D_f}=(-\infty,2)\xrightarrow{f}\mathrm{R_f}=(-\infty,2)\xrightarrow{g}\mathrm{R_{gf}}=(\ln 2,\infty)$ Therefore range of gf is $(\ln 2,\infty)$.	B1: ln2 B1: $(\ln 2,\infty)$	

Q4	Solution	Mark Scheme	Comments
(i)	$l:\mathbf{r}=\begin{pmatrix}9\\-5\\2\end{pmatrix}+\beta\begin{pmatrix}a\\b\\1\end{pmatrix}, \beta\in\mathbb{R}$ Since line is parallel to plane, the direction vector of line is perpendicular to the normal of the plane; $\begin{pmatrix}a\\b\\1\end{pmatrix}.\begin{pmatrix}3\\-4\\-1\end{pmatrix}=0$ $3a-4b=1$	B1: deduce direction perpendicular to normal B1: show mathematically	
(ii)(a)	$\begin{pmatrix}9+2\beta\\-5-2\beta\\2+\beta\end{pmatrix}.\begin{pmatrix}3\\-4\\-1\end{pmatrix}=-7$ $27+6\beta+20+8\beta-2-\beta=-7$ $13\beta=-52$ $\beta=-4$ The position vector of the point of intersection is $\begin{pmatrix}9\\-5\\2\end{pmatrix}+(-4)\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}1\\3\\-2\end{pmatrix}$	M1: sub eqn of line into eqn of plane A1	

Q4	Solution	Mark Scheme	Comments
(ii)(b)	Construct a line parallel to the normal of the plane and passing through point A: $r=\begin{pmatrix}9\\-5\\2\end{pmatrix}+\alpha\begin{pmatrix}3\\-4\\-1\end{pmatrix}, \alpha\in\mathbb{R}$. Subs. equation of line into that of plane: $\begin{pmatrix}9+3\alpha\\-5-4\alpha\\2-\alpha\end{pmatrix}.\begin{pmatrix}3\\-4\\-1\end{pmatrix}=-7$ $27+9\alpha+20+16\alpha-2+\alpha=-7$ $\alpha=-2$ The position vector of the foot of perpendicular is $\begin{pmatrix}9\\-5\\2\end{pmatrix}+(-2)\begin{pmatrix}3\\-4\\-1\end{pmatrix}=\begin{pmatrix}3\\3\\4\end{pmatrix}$ Distance = $\left\|\begin{pmatrix}9-3\\-5-3\\2-4\end{pmatrix}\right\|=\sqrt{6^2+(-8)^2+(-2)^2}=\sqrt{104}$ units	M1: form eqn of line through A and // to plane M1: sub eqn of line into plane A1: position vector M1: magnitude formula A1	
(ii)(c)	Distance of A to the point of intersection = $\left\|\begin{pmatrix}9-1\\-5-3\\2+2\end{pmatrix}\right\|=\sqrt{8^2+(-8)^2+(4)^2}=\sqrt{144}=12$ $\sin\theta=\frac{\sqrt{104}}{12}$ $\theta=58.2^{\circ}$	M1: angle formula A1	

Q5	Solution	Mark Scheme	Comments
(a)	Case (1): only 1 P used: _ _ P No. of required ways = $\binom{3}{2}\times 3!=18$ Case (2) : 2 P's used: _ P P No. of required ways = $\binom{3}{1}\times\frac{3!}{2!}=9$ Total required ways = 18 + 9 =27	M1: cases of 1P and 2P B1: 1 correct calculation A1	
(b)(i)	Required Number of ways = (9 – 1)! = 40320	B1	
(b)(ii)	There is only **1** way to arrange the female teachers in order of height. Next, position a male teacher in each of the slots between 2 female teachers: F _ F _ F _ F _ F No. of ways = 4! = 24	M1: 1 way to arrange in order of height A1	
(b)(iii)	No. of ways the male teachers are arranged among themselves = 4! = 24 No. of ways the female teachers are arranged among themselves = 5! = 120 Thera are 2 ways to arrange the two groups of male and female teachers. Total no. of ways = $2\times24\times120=5760$	M1: grouping and arranging within 1 group A1	

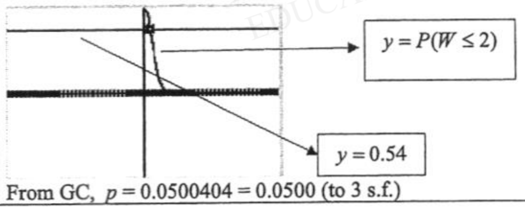
Q6	Solution	Mark Scheme	Comments
(i)(a)	$\frac{5+6}{36}=\frac{11}{36}$	B1	
(i)(b)	$\frac{5+8+7}{36}=\frac{5}{9}$	B1	

Q6	Solution	Mark Scheme	Comments
(ii)(a)	P(both are Aces | neither of the cards is Spade) $= \dfrac{\text{P(both Aces \& not Spade)}}{\text{P(not Spade)}}$ $= \dfrac{\binom{5}{2}}{\binom{30}{2}} = \dfrac{2}{87}$	M1: conditional probability with correct interpretation of intersection A1	
(ii)(b)	P('Queen not Diamond' AND 'Diamond not Queen') + P('Queen Diamond' AND 'Not Queen Not Diamond') $= \dfrac{\binom{9}{1}\binom{6}{1}}{\binom{36}{2}} + \dfrac{\binom{4}{1}\binom{17}{1}}{\binom{36}{2}} = \dfrac{61}{315}$	M1: 2 correct cases M1: calculate 1 case correctly A1	

Q7	Solution	Mark Scheme	Comments
(i)	Given: $\sum(x-12)=39,\ \sum(x-12)^2=2716.88$ Unbiased estimate of the population mean, $\bar{x}=\dfrac{39}{100}+12=12.39$ Unbiased estimate of the population variance, $s^2=\dfrac{1}{99}\left(\sum(x-12)^2-\dfrac{\left(\sum(x-12)\right)^2}{100}\right)$ $=27.2896=27.3$ (to 3 s.f.)	B1 B1	

Q7	Solution	Mark Scheme	Comments
(ii)	The researcher should carry out a one-tail test to test whether the data of incubation period for Mar has indeed increased as a few data points of 20 days raised an alarm. Let μ be the population mean incubation period for patients in Atopia. To test H_0: $\mu=10.8$ against H_1: $\mu>10.8$ at 1% significance level	B1: 1-tail and increase B1: all the symbols defined	
(iii)	Test statistic: Under H_0, $\overline{X}\sim \mathrm{N}\left(10.8,\dfrac{s^2}{100}\right)$ approximately by CLT since n is large $\Rightarrow Z=\dfrac{\overline{X}-10.8}{s/\sqrt{100}}\sim \mathrm{N}(0,1)$ approximately p-value $=0.00117<0.01$, we reject H_0. There is sufficient evidence at 1% significance level that the incubation period has increased.	B1: $\overline{X}\sim \mathrm{N}\left(10.8,\dfrac{s^2}{100}\right)$ approximately by CLT B1: p-value $=0.00117<0.01$ B1: reject with conclusion in context	
(iv)	To test H_0: $\mu=10.8$ against H_1: $\mu\neq 10.8$ 2-tail test at 1% significance level Under H_0, $Z=\dfrac{\overline{X}-10.8}{s/\sqrt{100}}\sim \mathrm{N}(0,1)$ Critical region to reject H_0:		

Q7	Solution	Mark Scheme	Comments
	By GC, $z_{calc} < -2.5758293$ or $z_{calc} > 2.5758293$ Since H_0 is rejected, $\frac{\bar{x}-10.8}{s/10} < -2.5758293$ or $\frac{\bar{x}-10.8}{s/10} > 2.5758293$ $\Rightarrow \bar{x} < 9.45$ or $\bar{x} > 12.1$ (to 3sf)	B1: z_{calc} inequalities with correct values M1: $z_{calc} = \frac{\bar{x}-10.8}{s/10}$ A1	

Q8	Solution	Mark Scheme	Comments
(i)	• The probability of each worker on sick leave is the same for all the 50 workers. • The event that a worker is sick is independent of the event that another worker is sick. 1st assumption can be invalid as some workers maybe more likely to fall sick than others. 2nd assumption can be invalid as if one worker is sick, he can spread his sickness to others.	B1 B1 B1: any reasonable answers	
(ii)	$W \sim B(50, p)$ $P(W \le 2) = 0.54$ $\binom{50}{0} p^0 (1-p)^{50} + \binom{50}{1} p^1 (1-p)^{49} + \binom{50}{2} p^2 (1-p)^{48} = 0.54$ $(1-p)^{50} + 50p(1-p)^{49} + 1225p^2(1-p)^{48} = 0.54$  From GC, $p = 0.0500404 = 0.0500$ (to 3 s.f.)	B1: $P(W \le 2) = 0.54$ and use of binomial formula B1: correct equation A1	

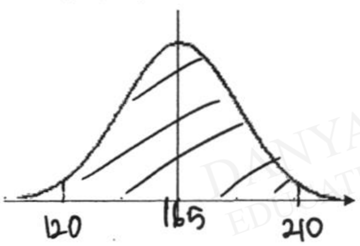
Q8	Solution	Mark Scheme	Comments
(iii)	Let X be the number of days (out of 60) for which there are at most 2 workers on sick leave. $X \sim B(60, 0.54)$ $E(X) = 60(0.54) = 32.4$	M1: $E(X) = np$ A1	
(iv)	Given $W \sim B(50, 0.0500404)$ $\Rightarrow E(W) = 50(0.0500404) = 2.50202$ and $Var(X) = 50(0.0500404)(0.9499596) = 2.37682$ Since $n = 60$ is large, by Central Limit Theorem, $\overline{W} \sim N(2.50202, \frac{2.37682}{60})$ approximately $P(\overline{W} \ge 2) = 0.994$	B1: correct variance B1: use of CLT with conditions stated B1: 0.994	

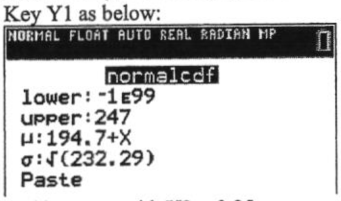
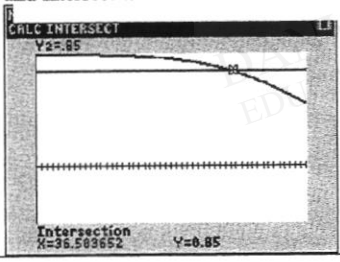
<table>
<tr><th>Q9</th><th>Solution</th><th>Mark Scheme</th><th>Comments</th></tr>
<tr><td>(i)</td><td>$\frac{\binom{4}{1}\binom{3}{3}}{\binom{7}{4}} = \frac{4}{35}$</td><td>B1</td><td></td></tr>
<tr><td>(ii)</td><td>
<table>
<tr><td>x</td><td>1</td><td>2</td><td>3</td><td>4</td></tr>
<tr><td>$P(X = x)$</td><td>$\frac{\binom{4}{1}\binom{3}{3}}{\binom{7}{4}} = \frac{4}{35}$</td><td>$\frac{\binom{4}{2}\binom{3}{2}}{\binom{7}{4}} = \frac{18}{35}$</td><td>$\frac{\binom{4}{3}\binom{3}{1}}{\binom{7}{4}} = \frac{12}{35}$</td><td>$\frac{1}{35}$</td></tr>
</table>
</td><td>B1: 1 correct

B1: 2 correct

B1: all correct</td><td></td></tr>
</table>

Q9	Solution	Mark Scheme	Comments
(iii)	$E(X)=1.\frac{4}{35}+2.\frac{18}{35}+3.\frac{12}{35}+4.\frac{1}{35}=\frac{16}{7}$ $E(X^2)=1^2.\frac{4}{35}+2^2.\frac{18}{35}+3^2.\frac{12}{35}+4^2.\frac{1}{35}=\frac{40}{7}$ $\text{Var}(X)=E(X^2)-[E(X)]^2=\frac{40}{7}-\left(\frac{16}{7}\right)^2=\frac{24}{49}$	B1 B1: calculate $E(X^2)$ and show clearly	
(iv)	Let X_1, X_2, X_3 be the correct guesses by 3 listeners respectively. $P(X_1+X_2+X_3=6)$ $= P(X_1=1)\cdot P(X_2=2)\cdot P(X_3=3)\times 3! +$ $P(X_1=2)\cdot P(X_2=2)\cdot P(X_3=2)+$ $P(X_1=1)\cdot P(X_2=1)\cdot P(X_3=4)\times 3$ $=\left(\frac{4}{35}\right)\left(\frac{18}{35}\right)\left(\frac{12}{35}\right)\times 6+\left(\frac{18}{35}\right)^3+\left(\frac{4}{35}\right)^2\left(\frac{1}{35}\right)\times 3$ $= 0.258$ One assumption we have made is that the guesses of listeners are independent of each other.	M1: correct cases M1: calculate 1 case correct A1 B1	
(v)	Required probability $=35\times\left(\frac{1}{35}\right)^3$ $=\frac{1}{1225}$ <u>Alternatively</u> $\frac{\binom{7}{4}}{\binom{7}{4}^3}=\frac{1}{1225}$	M1: power 3 to account for same guesses. A1	

Q10	Solution	Mark Scheme	Comments
(i)	$165 \pm 3(15)$ gives a range of (120,210). Let P be the mass of phone body in grams. $P \sim N(165, 15^2)$  *(Note: 99.7% of the data will lie within 3 standard deviations, thus the shading should take up almost the whole area)*	B1: symmetrical bell shape with mean 165 B1: 210 and 120 labels	
(ii)	$P_1-P_2 \sim N\left(0,2\left(15^2\right)\right)=N(0,450)$ $P(\|P_1-P_2\|<0.5)$ $=P(-0.5<P_1-P_2<0.5)$ $=0.0188$	B1: $P_1-P_2 \sim N(0,450)$ M1: set up as modulus or from -0.5 to 0.5 A1	
(iii)	Let B be the mass of a battery, i.e., $B=0.18P$ $B=0.18P$ $Y=P+B=1.18P \sim N(194.7, 313.29)$ P (ideal mass) $=P(140<Y<170)$ $=0.0804$ (to 3 sf, by *GC*)	B1: correct mean for phone with battery B1: correct variance for phone with battery A1	Note that we cannot find B and P distribution separately and add together since they are dependent variables. We have to combine into $1.08P$ first.

(iv)	Let X be the random variable for the total mass of the phone body, battery and travel charger $X = Y + q$, where q is a constant $X \sim \mathrm{N}(194.7+q, 232.29)$ $P(X < 247) = 0.85$ $\mathrm{P}(Z < \frac{247-(194.7+q)}{\sqrt{232.29}}) = 0.85$ By GC, $\frac{52.3-q}{\sqrt{232.29}} = 1.03643338$ $q = 36.504 = 36.5$ g *Alternatively,* Use GC graph to solve Key Y1 as below:  and intersect with Y2 = 0.85  	B1: $X \sim \mathrm{N}(194.7+q, 232.29)$ M1: standardisation M1: invNorm A1	GC alert! Even if not using GC graph to solve, can use it as checking mechanism.