ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Mathematics Paper 1 (100 marks)

14 Sept 2020

3 hours

Additional Material(s): List of Formulae (MF26)

CANDIDATE NAME

CLASS

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
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Question number	Marks
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This document consists of 18 printed pages and 2 blank pages.

[Turn Over

1 A convergent geometric progression G has nth term denoted by u_n. It has a positive first term and common ratio r. Another geometric progression H of positive numbers has nth term denoted by v_n and common ratio $\frac{1}{R}$. If $0 < r < R < 1$, show that a new sequence whose nth term is $\ln(u_n v_n)$ is an arithmetic progression. [2]

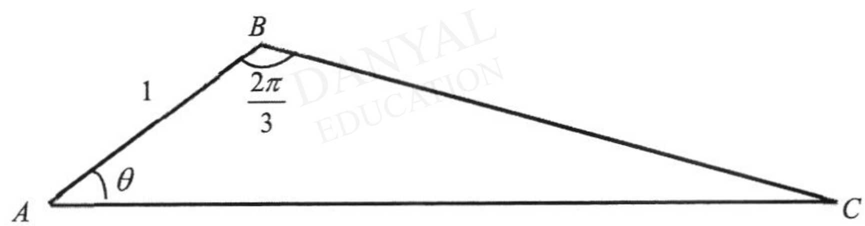
Determine if this sequence is decreasing or increasing. Justify your answer. [2]

2 It is given that $f(x) = 2kx^3 + (5k-2)x^2 + (k-5)x + 3 - 2k$, where k is a real constant. The curve with equation $y = f(x)$ has only 1 x-intercept at $x = \frac{1}{2}$.

(i) Find the exact range of values of k. [3]

(ii) What can be said about the other two solutions to the equation $f(x) = 0$? [1]

3



In the triangle ABC, $AB = 1$, angle $BAC = \theta$ radians and angle $ABC = \frac{2\pi}{3}$ radians (see diagram).

(i) Show that $AC = \dfrac{\sqrt{3}}{\sqrt{3}\cos\theta - \sin\theta}$. [3]

(ii) Given that θ is a sufficiently small angle, show that

$$AC \approx 1 + a\theta + b\theta^2,$$

for constants a and b to be determined exactly. [3]

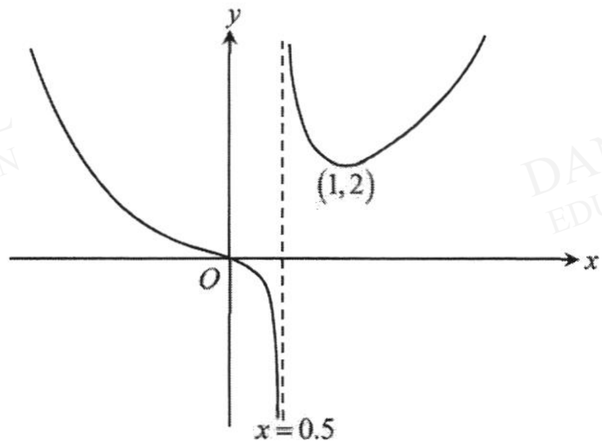
4 **(a)** The curve C has equation $y=\dfrac{x^2-5x+7}{4-2x}$.

By expressing the equation of C in the form $y=A\left[(3-x)+\dfrac{1}{(3-x)+B}\right]$,

where A and B are constants to be found, find a sequence of transformations

which transforms the graph of C on to the graph of $y=x+\dfrac{1}{x-1}$. [4]

(b)



The diagram shows the graph of $y=\mathrm{f}(x)$ with an asymptote $x=0.5$. The curve passes through the origin O and has only one stationary point at $(1,2)$.

Sketch the graph of $y=\dfrac{1}{\mathrm{f}(x)}$, showing clearly the equations of any asymptotes and the coordinates of any turning points and x–intercepts. [3]

5 **(i)** Express $\dfrac{4}{4r^2+16r+15}$ as $\dfrac{A}{2r+3}+\dfrac{B}{2r+5}$, where A and B are constants to be determined. [1]

The sum $\displaystyle\sum_{r=1}^{n}\frac{4}{4r^2+16r+15}$ is denoted by S_n.

(ii) Find an expression for S_n in terms of n. [3]

(iii) Find the smallest value of n for which S_n is within 10^{-3} of the sum to infinity. [3]

(iv) Using the result in **(ii)**, find $\displaystyle\sum_{r=10}^{2n}\frac{2}{4r^2-1}$ in terms of n. [3]

6 Referred to the origin O, points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Point C lies on BA produced such that $BA : AC = \lambda : \mu$, where $\lambda, \mu > 0$.

(i) Show that $\overrightarrow{OC}$ is given by $\frac{\lambda+\mu}{\lambda}\mathbf{a} - \frac{\mu}{\lambda}\mathbf{b}$. [1]

(ii) It is given that $\mathbf{a}$ is a unit vector, $|\mathbf{b}| = 3$ and angle AOB is $\cos^{-1}\left(\frac{2}{3}\right)$. Find the ratio $BA : AC$ such that C, which is on the line AB, is nearest to O. [4]

The position vector $\mathbf{v}_k$ is such that $\mathbf{v}_k = \mathbf{a} + k(\mathbf{b} - \mathbf{a})$ where k is a positive integer. If $\overrightarrow{OS} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \ldots + \mathbf{v}_n$,

(iii) find $\overrightarrow{OS}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and n. Simplify your answer. [2]

(iv) If $\lambda = 2$ and $\mu = 3$, find the area of triangle OCS in the form $k|\mathbf{a} \times \mathbf{b}|$, where k is in terms of n. [3]

7 **(a)** The function f is such that $\mathrm{f} : x \mapsto -2x^2 + 4x + 1, \quad x \in \mathbb{R}, x \leq a$.

State the greatest value of a such that f^{-1} exists and for this value of a, find f^{-1} in a similar form. [4]

(b) The function g is a strictly decreasing and continuous function such that $\mathrm{g}(x) = -\mathrm{g}(-x)$ for $x \in \mathbb{R}$. The coordinates of certain points on the curve of $y = \mathrm{g}(x)$ are as follows:

x	-6	-5.7	-5	-4	-3.2	-3	-1	0	2	4.3
y	7	$4\sqrt{3}$	6	$3\sqrt{3}$	4.3	4	$2\sqrt{3}$	0	-3.7	-5.5

(i) State the value of $\mathrm{g}^{-1}(-6)$. [1]

Another function h is defined by $\mathrm{h} : x \mapsto 4\sin 2x$ for $0 \leq x \leq \frac{\pi}{2}$.

(ii) Find the exact range of the composite function gh. Find the range of values of x such that the composite function gh satisfies the inequality $\left|\mathrm{g}^{-1}\mathrm{h}(x) + 3\right| < 2$. Leave your answer in exact form. [4]

7 **(c)** The function k is defined by

$$\mathrm{k}: n \mapsto \begin{cases} \sqrt{n}+\dfrac{n}{2}, & \text{for odd positive integer } n \\ n, & \text{for even positive integer } n. \end{cases}$$

Explain clearly whether the composite function k^2 exists. [1]

8 **Do not use a calculator in answering this question.**

(a)(i) Solve the equation $w^2 = 3-4\mathrm{i}$, giving your answers in cartesian form $a+\mathrm{i}b$. [4]

(ii) Hence find the roots of the equation $z^2 - 4\mathrm{i}z + 4\mathrm{i} - 7 = 0$, giving your answers in cartesian form $p+q\mathrm{i}$. [2]

(b) For positive integer n, a complex number z is such that $|z|^n = \dfrac{1}{\sqrt{2}}$. By considering the conjugate of $1+2z^{2n}$, show that $\dfrac{2z^n}{1+2z^{2n}}$ is a real number. [4]

9 **(a)(i)** Find $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\sin(x^3+2)\right]$. [1]

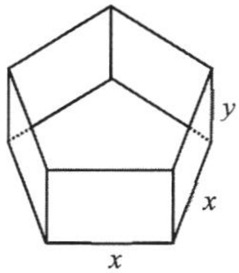
(ii) Find $\displaystyle\int 2x^5\cos(x^3+2)\,\mathrm{d}x$. [2]

(b)(i) Using an algebraic approach, solve the inequality $\dfrac{x-16}{x^2-16} \le 1$. [3]

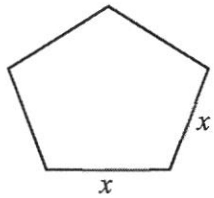
(ii) Find $\displaystyle\int \left(\frac{x-16}{x^2-16}-1\right)\mathrm{d}x$. [3]

(iii) Hence find the exact value of $\displaystyle\int_{-1}^{1}\left|\frac{x-16}{x^2-16}-1\right|\mathrm{d}x$ in the form of $\ln\dfrac{a}{b}$, where a and b are real constants. [3]

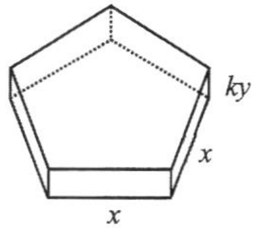
10



Container



Cross-section of Container



Lid

Crayole, a handicraft company, requires a container of negligible thickness to hold 200 cm^3 of party slime. The cross-section of the container is a regular pentagon with sides x cm and the height of the container is y cm. The sides of its lid is x cm and has a depth ky cm, where $0 < k \leq 1$ (see diagram).

It is known that the area of a regular pentagon with sides x cm is given by $\frac{ax^2}{4}$, where a is a positive fixed constant.

(i) Use differentiation to find, in terms of a and k, the exact value of x which gives a minimum total external surface area of the container and the lid. [6]

(ii) Find the ratio $\frac{y}{x}$ in terms of a and k in this case, simplifying your answer. [2]

(iii) Find the range of values of $\frac{y}{x}$ in terms of a. [2]

(iv) Find the exact value of k, in terms of a, if the company requires the sides of the container to be $\frac{3}{4}$ of its height. [2]

11 **(a)** At the beginning of January 2010, Mr Wong's fish farm has 40 000 fish breeding in a large pond. In the period between January and December each year, the number of fish increases by 9%. At the end of December each year, 5400 fish are harvested and sold at the fish markets for $6 per fish. No fish died or are being poached while he is in business.

(i) Find the number of fish just after the nth harvest, giving your answer in the form $A - B(1.09)^n$, where A and B are constants to be determined. [3]

(ii) Find the earliest month and year in which there will be no more fish in the pond. [2]

Mr Wong has been thinking of retiring since January 2010. Every December just after the harvest, a nearby fish farm offers to buy over Mr Wong's business by paying $8 per fish for his remaining stock.

(iii) If he wishes to retire by selling off his fish farm business with the maximum gain after the nth harvest, find the value of n. [3]

(b) White-tailed deer can be found in forests of Southern Canada. A wildlife biologist is investigating the change of a population of white-tailed deer of size x (thousands) at time t (years). He observed that on average, their birth rate is 4000 per year and their death rate is proportional to the square of the population present.

(i) When the population was 5000, the rate of increase of the population at that instant was 1750 per year. Assuming that x and t are continuous variables, form a differential equation relating x and t. [2]

(ii) There were 2000 deer in the forest initially. Solve this differential equation to obtain x as an exact function of t. [4]

End of Paper

ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Mathematics Paper 2 (100 marks)

18 Sept 2020

3 hours

Additional Material(s): List of Formulae (MF 26)

CANDIDATE NAME

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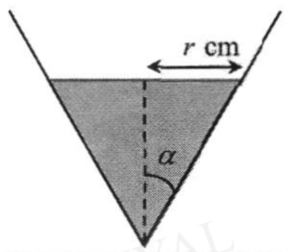
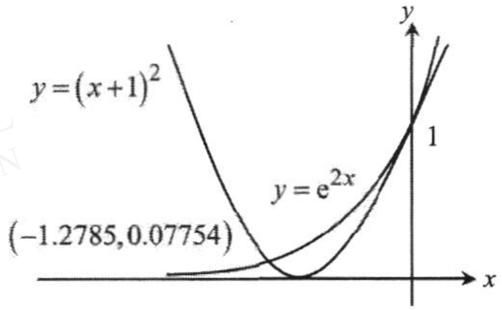
	Section A: Pure Mathematics [40 marks]	
1	The complex number z and w are given by $z=2\mathrm{e}^{\mathrm{i}\theta}$, where $\frac{3\pi}{4}<\theta\le\pi$, and $w=1+\sqrt{3}\mathrm{i}$. Express $w-z$ in exponential form $re^{i\alpha}$, where both r and α are in terms of θ and $-\pi<\alpha\le\pi$.	[3]
2	**(a)** A curve is defined by the parametric equations $$x=\frac{1}{t},\qquad y=t^2,\qquad \text{for } 0<t<1.$$ Show that the equation of the normal to the curve at the point $P\left(\frac{1}{p},p^2\right)$ is $$2p^4y-px=2p^6-1$$ Hence, using an algebraic method, show that the normal at P cuts the curve exactly once.	[6]
	(b) 	
	A cone of semi-vertical angle α, where $\tan\alpha=\frac{1}{3}$, is held with its axis vertical and vertex downwards (see diagram). At the beginning of an experiment, it is filled with 300 cm^3 of liquid water. Water runs out of a small hole at the vertex at a constant rate of 0.2 cm^3 per second. At time t minutes after the start, the radius of the water surface is r cm (see diagram). Find the rate at which the depth of the liquid is decreasing 5 minutes after the start of the experiment. [The volume of a cone of base radius r and height h is given by $V=\frac{1}{3}\pi r^2h$.]	[5]
3	A drone carrying a bomb departs from a point $A(-1,1,-3)$. It flies in a direction of $-a\mathbf{i}+\mathbf{j}+2a\mathbf{k}$, where $a>0$, across a lake to a target location. It is given that the surface of the lake is part of a plane with equation $x-z=2$.	
	(i) Determine the value of a if the path of the drone makes an angle of $\frac{\pi}{3}$ radians with the surface of the lake.	[3]
	It is given that $a=2$.	
	(ii) A missile is launched to intercept the drone. It moves in a path with equation $\frac{x-10}{2}=z,\ y=m$, where m is a real constant. Given that the missile is successful in intercepting the drone, find the point of interception.	[3]

(iii) At the point of interception, a piece of the drone falls perpendicular to the lake and meets the surface of the lake at N. Find the position vector of N. [3]

(iv) Denoting the line that contains the drone's path as L_1 and $L_1{}'$ being the reflection of L_1 in the surface of the lake, find a vector equation of $L_1{}'$. [3]

4 **(a)** Find $\int \left(\cot^6 2x + \cot^4 2x - \sin\frac{x}{2}\sin\frac{3x}{2} \right) \mathrm{d}x$. [3]

(b) The diagram shows the graphs of $y = \mathrm{e}^{2x}$ and $y = (x+1)^2$.

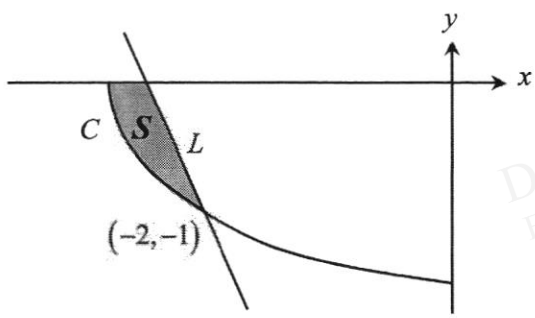


R is the finite region bounded by the two curves $y = \mathrm{e}^{2x}$ and $y = (x+1)^2$. Find the volume of the solid formed when R is rotated through 2π radians about the y-axis, giving your answer correct to 4 decimal places. [3]

(c) A curve C has parametric equations

$$x = 2\left(\sin\frac{t}{2} - \cos\frac{t}{2}\right), \quad y = \sin\frac{t}{2} + \cos\frac{t}{2}, \quad \text{for } -\frac{3\pi}{2} \le t \le -\frac{\pi}{2}.$$

The line L with equation $y = -2x - 5$ meets the curve C at the point $(-2,-1)$. S is the region enclosed by line L, curve C and the x-axis as shown in the diagram below.



(i) Show that the x-intercept of curve C is $-2\sqrt{2}$. [2]

(ii) Find the exact area of S. [6]

Section B: Statistics [60 marks]

5 Elaine always receives two \$2 notes, two \$5 notes and one \$20 note from her parents as her monthly allowance. She decides to select two of these notes to contribute to Community Chest for the month of August. The total value of these notes is denoted by \$$C$.

(i) Determine the probability distribution of C. [3]

(ii) Find the expected amount of money that she will donate in August and the variance of C. [2]

In September, Elaine decides to contribute to Community Chest again.

(iii) Find the probability that her total amount of donations for August and September is at least \$12. [2]

6 A group of 300 students are asked whether they own any earbuds, laptops or games machines.

90 students own a pair of earbuds
177 students own a laptop
100 students own a games machine

Events A, L and G are defined as follows:

A: a randomly chosen student owns a pair of earbuds.
L: a randomly chosen student owns a laptop.
G: a randomly chosen student owns a game machine.

It is given that events A and G are independent events. It is also given that 35 students own a laptop and a games machine and that 20 students own all the three gadgets.

(i) Find the probability that a student selected at random owns either a games machine or a pair of earbuds but not both. [2]

(ii) A student selected at random owns a games machine. Find the probability that the student owns exactly two of the gadgets. [2]

(iii) Find the greatest and least value of $\mathrm{P}(A \cap L \cap G')$. [3]

7 The masses, in grams, of mangoes have the distribution $\mathrm{N}(200, 30^2)$ and the masses, in grams, of oranges have the distribution $\mathrm{N}(\mu, \sigma^2)$.

(i) Let A be the average mass of a mango and two oranges. Given that $\mathrm{P}(A < 162) = \mathrm{P}(A > 206) = 0.06$, find μ. Hence form an equation involving σ and solve for σ. [5]

(ii) Find probability that the total mass of 3 mangoes differs from 3 times the mass of an orange by at most 136 grams. [3]

(iii) Mangoes are sold at \$26 per kilogram. Find the probability that the total selling price of 6 mangoes exceeds \$29.50. [2]

8 The boss of a hair salon chain claims that the waiting time, on average, is 15 minutes. Jeremy believes that the waiting time quoted by the hair salon chain is understated. He carries out a survey to investigate this by asking the waiting time for 110 customers. The waiting time are summarised by

$$\sum t = 1900, \text{ and } \sum (t-15)^2 = 25400.$$

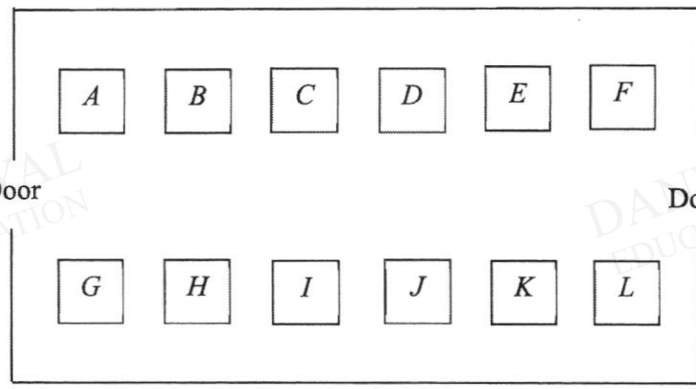
(i) Find the unbiased estimate of the population mean. Leave your answers correct to 3 decimal places. Show that the unbiased estimate of the population variance is given by 227.815. [2]

(ii) Carry out a test on Jeremy's belief at the 5% significance level, stating a necessary assumption for the test. [5]

(iii) Kalai did another test to determine whether the mean waiting time differs from 15 minutes using another random sample. The waiting times of n [4]

randomly chosen customers, where n is large, were recorded, and their mean and standard deviation are 15.6 minutes and 2.5 minutes respectively. Given that the null hypothesis is rejected at the 5% level of significance, find the least value of n.

9 A rectangular barrack with a door at each end, has 12 beds marked $A, B, C, \dots, L$ as shown in the diagram below. Johnson and 11 other soldiers take 1 bed each.



(i) Find the number of different sleeping arrangements possible if none of Johnson and 2 other particular soldiers are adjacent to each other, and all three of them are on the same side of the barrack. [2]

One afternoon, 9 particular soldiers went out to have lunch. There were 3 identical round tables at the dining venue.

(ii) How many ways can 9 soldiers be seated if there must be at least two soldiers at each table? [4]

Johnson was playing with a deck of 10 cards. When the cards are arranged in a certain manner, it forms the word SUCCESSFUL.

(iii) Find the number of ways he can arrange the 10 cards such that not all the letter "S" are together. [2]

(iv) He wishes to replace his existing handphone 4-letter codeword using letters from the word SUCCESSFUL, how many different options would he have. [4]

10 A ferry company operates a 9am ferry, with a passenger capacity of 108, from Island X to Island Y daily. To maximise profits, 113 tickets are sold online each day because it was found that on average, p% of customers who have purchased a ticket do not turn up.

(i) It is known that there is a probability of 0.012 that at most 1 customer will not turn up for the ferry. Write down an equation in terms of p, and hence find p correct to 3 decimal places. [3]

It is now given that $p = 6$.

(ii) Find the probability that there are no empty seats if every customer who turns up get a seat on the 9am ferry. [3]

(iii) Find the probability that, in a randomly chosen week, every customer who turns up gets a seat on the 9am ferry on at least 5 days of the week. [2]

(iv) Find the maximum number of tickets that could be sold so that the probability of a customer not getting a seat on the ferry is no more than 0.01. [3]

	(v) A period of forty days are randomly selected. Find the probability that the mean daily number of customers who purchased a ticket and managed to turn up during this period is at most 106.	[2]

End of Paper

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MATHEMATICS **9758**

H2 Mathematics Paper 1 (100 marks) **14 Sept 2020**

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1 A convergent geometric progression G has nth term denoted by u_n. It has a positive first term and common ratio r. Another geometric progression H, of positive numbers, has nth term denoted by v_n and common ratio $\frac{1}{R}$. If $0<r<R<1$, show that a new sequence whose nth term is $\ln(u_n v_n)$ is an arithmetic progression. [2]

Determine if this sequence is decreasing or increasing. Justify your answer. [2]

Solution

$\ln(u_n v_n) - \ln(u_{n-1} v_{n-1})$

$=\ln\left(\frac{u_n v_n}{u_{n-1} v_{n-1}}\right)$

$=\ln\left(\left(\frac{u_n}{u_{n-1}}\right)\left(\frac{v_n}{v_{n-1}}\right)\right)$

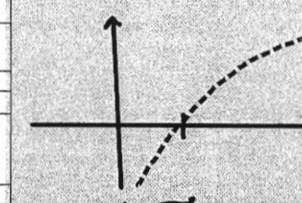
$=\ln\left(r\times\frac{1}{R}\right)$

$=\ln\left(\frac{r}{R}\right)=$ constant

Since r and R are both constants as G and H are both G.P.
Since difference between consecutive terms of the new sequence is constant, it is an arithmetic progression.

[redacted]

Commented [KSM1]: 1)The proof should be for the general term. 2)Getting $u_n v_n$ in the form of AP is not a proper way to show.



⇒ ln r < ln R
⇒ ln r − ln R < 0
⇒ ln(r/R) < 0

2 It is given that $f(x)=2kx^3+(5k-2)x^2+(k-5)x+3-2k$, where k is a real constant. The curve with equation $y=f(x)$ has only 1 x-intercept at $x=\frac{1}{2}$.

(i) Find the exact range of values of k. [3]

(ii) What can be said about the other two solutions to the equation $f(x)=0$? [1]

Solution

(i) $2kx^3+(5k-2)x^2+(k-5)x+3-2k=\left(x-\frac{1}{2}\right)(2kx^2+4x+4k-6)=0$

[redacted]

6k − 2

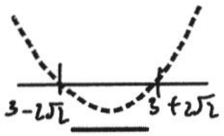
D < 0

−kx² + Ax²

(A − x)

(∵ no roots)

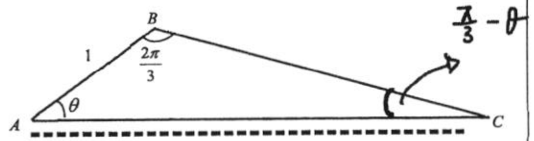
Commented [TCK2]: Candidates often do not get the quadratic factor correctly because they used long division method which proves tedious. Rather, by employing a little observatio the coefficient of x^2 and constant could be foun easily.



	Since it has only 1 real root,	
	$kx^2+(3k-1)x+2k-3=0$ has no real roots.	
	$(3k-1)^2-4(k)(2k-3)<0$	
	$9k^2-6k+1+12k-8k^2<0$	
	$k^2+6k+1<0$	
	$(k+3)^2-8<0$	
	$(k+3+2\sqrt{2})(k+3-2\sqrt{2})<0$	
	$-2\sqrt{2}-3<k<2\sqrt{2}-3$	
	(ii) Since the coefficients of the equation are all real numbers, the other two roots are complex numbers and are conjugate of each other.	

Commented [TCK3]: Unfortunately, there are candidates who still process this as $k<-3\pm\sqrt{8}$.

Commented [TCK4]: It is not sufficient to say that the two solutions will be not real. Interestingly, many candidates used the word 'imaginary' rather than the appropriate word 'complex'.

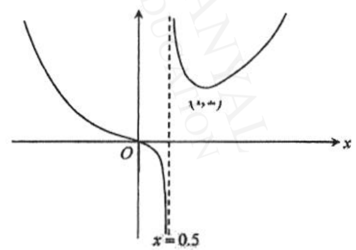
3	 In the triangle ABC, $AB=1$, angle $BAC=\theta$ radians and angle $ABC=\frac{2\pi}{3}$ radians (see diagram).	
	(i) Show that $AC=\dfrac{\sqrt{3}}{\sqrt{3}\cos\theta-\sin\theta}$.	[3]
	(ii) Given that θ is a sufficiently small angle, show that $AC\approx1+a\theta+b\theta^2$, for constants a and b to be determined exactly.	[3]
	Solution	
	(i) angle $ACB=\pi-(\theta+\frac{2}{3}\pi)$ $=\frac{\pi}{3}-\theta$	
	By sine rule, $\dfrac{AC}{\sin\left(\frac{2}{3}\pi\right)}=\dfrac{1}{\sin\left(\frac{\pi}{3}-\theta\right)}$	
	$\dfrac{AC}{\sqrt{3}/2}=\dfrac{1}{\sin\frac{\pi}{3}\cos\theta-\cos\frac{\pi}{3}\sin\theta}$	
	$\dfrac{AC}{\sqrt{3}/2}=\dfrac{1}{\frac{\sqrt{3}}{2}\cos\theta-\frac{1}{2}\sin\theta}$	

$\frac{\pi}{3}-\theta$

use of sine rule

Commented [AA5]: A few students considered a different pairs of ratios and struggled to get the answer.

Commented [AA6]: Generally, this part is done well. A few cases of sign mistake.

	$AC=\dfrac{\sqrt{3}}{\sqrt{3}\cos\theta-\sin\theta}$ (shown)	
	(ii) [redacted] $\cos\theta\approx1-\dfrac{\theta^2}{2}$, $\sin\theta\approx\theta$	
	$AC=\dfrac{\sqrt{3}}{\sqrt{3}\cos\theta-\sin\theta}$	
	$AC=\dfrac{1}{\cos\theta-\frac{1}{\sqrt{3}}\sin\theta}$	
	$\approx\left(1-\dfrac{\theta^2}{2}-\dfrac{\theta}{\sqrt{3}}\right)^{-1}$	
	$=\left(1-\left(\dfrac{\theta}{\sqrt{3}}+\dfrac{\theta^2}{2}\right)\right)^{-1}$	
	$=1+\left(\dfrac{\theta}{\sqrt{3}}+\dfrac{\theta^2}{2}\right)+$ [redacted] $+\dfrac{\theta^2}{2}\Big)^2+\cdots$ (binomial expansion)	
	$\approx1+\left(\dfrac{\theta}{\sqrt{3}}+\dfrac{\theta^2}{2}\right)+\dfrac{\theta^2}{3}+\cdots$	
	$\approx1+\dfrac{1}{\sqrt{3}}\theta+\dfrac{5}{6}\theta^2$ (shown)	
	$\therefore a=\dfrac{1}{\sqrt{3}}$; $b=\dfrac{5}{6}$	
4	**(a)** The curve C has equation $y=\dfrac{x^2-5x+7}{4-2x}$.	
	By expressing the equation of C in the form $y=A\left[(3-x)+\dfrac{1}{(3-x)+B}\right]$, where A and B are constants to be found, find a sequence of transformations which transforms the graph of C on to the graph of $y=x+\dfrac{1}{x-1}$.	[4]
	(b) 	

$(1+x)^n$: Binomial expansion.

must be extracted as x.

Commented [AA7]: Most of the students were able to replace cos and sin by their small angle approximations. However they failed to recognize it as binomial expansion.

Commented [AA8]: Most students used the general expansion formula rather than identifying it as standard expansion.

Commented [AA9]: Careless calculation mistake.

Commented [KSM10]: Many mistaken the transformation to be from $y=x+\frac{1}{x-1}$ to graph of C.
So many get the scale factor wrong. Here you should replace y by y/2, so scale factor should be 2.

The diagram shows the graph of $y=f(x)$ with an asymptote $x=0.5$. The curve passes through the origin O and has only one stationary point at $(1,2)$.

Sketch the graph of $y=\frac{1}{f(x)}$, showing clearly the equations of any asymptotes and the coordinates of any turning points and x–intercepts. [3]

Solution

(a)

$C: \frac{x^2-5x+7}{4-2x} = \frac{x^2-5x+7}{2(2-x)} = \frac{1}{2}\left[(3-x)+\frac{1}{(3-x)+B}\right]$

$A=\frac{1}{2}, 3+B=2 \Rightarrow B=-1$

$\therefore C: 2y=\left[(3-x)+\frac{1}{(3-x)-1}\right]$

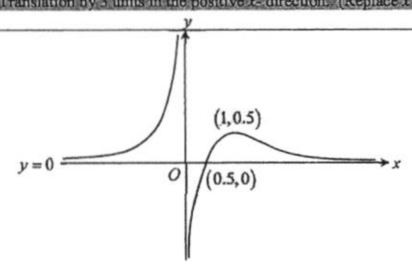
Transform C to $y=x+\frac{1}{x-1}$

1
3 → Translation – x direction
2 Reflection in the y-axis.

1. Scaling parallel to the y-axis by scale factor 2. (Replace y by y/2)
2. Reflection in the y – axis. (Replace x by – x)
3. Translation by 3 units in the positive x- direction. (Replace x by x – 3)

Commented [KSM11]: If translate then reflect, then you should do 1)Translation of 3 units in the negative x-direction 2)Reflection in y-axis

(b)



5 (i) Express $\frac{4}{4r^2+16r+15}$ as $\frac{A}{2r+3}+\frac{B}{2r+5}$, where A and B are constants to be determined. [1]

The sum $\sum_{r=1}^{n}\frac{4}{4r^2+16r+15}$ is denoted by S_n.

(ii) Find an expression for S_n in terms of n. [3]

(iii) Find the smallest value of n for which S_n is within 10^{-3} of the sum to infinity. [3]

(iv) Using the result in (ii), find $\sum_{r=10}^{2n}\frac{2}{4r^2-1}$ in terms of n. [3]

Solution

(i) Let $\frac{4}{4r^2+16r+15}=\frac{4}{(2r+3)(2r+5)}=\frac{A}{2r+3}+\frac{B}{2r+5}$

$4=A(2r+5)+B(2r+3)$

When $r=-\frac{3}{2}$, $4=A(-3+5)\Rightarrow A=2$

When $r=-\frac{5}{2}$, $4=B(-5+3)\Rightarrow B=-2$

$\frac{4}{4r^2+16r+15}=\frac{2}{2r+3}-\frac{2}{2r+5}$

(ii) $S_n=\sum_{r=1}^{n}\frac{4}{4r^2+16r+15}=\sum_{r=1}^{n}\frac{4}{(2r+3)(2r+5)}$

$=\sum_{r=1}^{n}\left(\frac{2}{2r+3}-\frac{2}{2r+5}\right)$

$=\left[\begin{array}{l}\frac{2}{5}-\frac{2}{7}\\+\frac{2}{7}-\frac{2}{9}\\+\frac{2}{9}-\frac{2}{11}\\+\ldots\\+\frac{2}{2n+1}-\frac{2}{2n+3}\\+\frac{2}{2n+3}-\frac{2}{2n+5}\end{array}\right]$

$=\frac{2}{5}-\frac{2}{2n+5}$

(iii) $|S_n-S_\infty|<10^{-3}$

$\left|\left(\frac{2}{5}-\frac{2}{2n+5}\right)-\frac{2}{5}\right|<10^{-3}$

$\frac{2}{2n+5}<10^{-3}$

$2n+5>2000$

$n>997.5$

$\therefore$ Smallest value of $n=998$

(iv) $\sum_{r=10}^{2n}\frac{2}{4r^2-1}=\frac{1}{2}\sum_{r+2=10}^{r+2=2n}\frac{4}{4(r+2)^2-1}$ (replace r by $r+2$)

$=\frac{1}{2}\sum_{r=8}^{2n-2}\frac{4}{4r^2+16r+15}$

$=\frac{1}{2}\left[\sum_{r=1}^{2n-2}\frac{4}{4r^2+16r+15}-\sum_{r=1}^{7}\frac{4}{4r^2+16r+15}\right]$

$\frac{2}{4(r+2)^2-1}=\frac{2}{4(r^2+4r+4)-1}=\frac{2}{4r^2+16r+16-1}$

$=\frac{2}{4r^2+16r+15}$

Commented [KML12]: Careless Mistake
Some students were careless and interchanged the values of A and B

Commented [KML13]: Presentation of Answer
Majority of students didn't put in the brackets.

Commented [KML14]: Careless Mistake
Some students didn't put in the "+" sign in front each row. Some didn't cancelled out the terms – 2/11 and 2/(2n+1).

Commented [KML15]: Presentation of Answer
It can also be written as
$S_\infty - S_n < 10^{-3}$ as $S_\infty > S_n$
since each of the terms are positive.
Common mistakes:
1) As $n\to\infty$, $S_\infty \to \frac{2}{5}$ instead $S_n \to \frac{2}{5}$
2) $S_n < 10^{-3} S_\infty$
3) $S_n < S_\infty - 10^{-3}$. It should be $S_n > S_\infty - 10^{-3}$.
4) Some students put in equality sign, i.e. $S_n = S_\infty - 10^{-3}$ or $S_n = S_\infty \pm 10^{-3}$ and obtained $n=997.5$ which they approximated to 998. This is obviously wrong as question asked for smallest value of n.

Commented [KML16]: Presentation of Answer & Careless Mistake
Students need to explain what replacement they are making for r. Some students replace r by r+2 and change the lower limit but not the upper limit. Some replace r by r+2 in the expression but replace r by r-2 in the limits.

Commented [KML17]: Presentation of Answer
Students should express the function in r to be the same as that in (ii) before using the result.

Commented [KML18]: Presentation of Answer
Students should show this step before substituting the result from (ii).

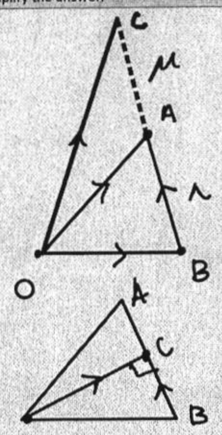
$\sum_1 = \frac{2}{5} - \frac{2}{2n+5}$

$\sum_1^{2n-2} = \frac{2}{5} - \frac{2}{2(2n-2)+5} = \frac{2}{5} - \frac{2}{4n-1}$

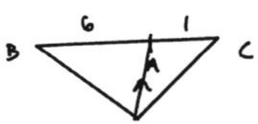
$\sum_1^{7} = \frac{2}{5} - \frac{2}{2(7)+5} = \frac{2}{5} - \frac{2}{19}$

	$= \frac{1}{2}\left[\left(\frac{2}{5} - \frac{2}{2(2n-2)+5}\right) - \left(\frac{2}{5} - \frac{2}{2(7)+5}\right)\right]$	
	$= \frac{1}{19} - \frac{1}{4n+1}$	
6	Referred to the origin O, points A, B and C have position vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Point C lies on BA produced such that $BA : AC = \lambda : \mu$, where $\lambda, \mu > 0$.	
	(i) Show that $\overrightarrow{OC}$ is given by $\frac{\lambda+\mu}{\lambda}\mathbf{a} - \frac{\mu}{\lambda}\mathbf{b}$.	[1]
	(ii) It is given that $\mathbf{a}$ is a unit vector, $\lvert\mathbf{b}\rvert = 3$ and angle AOB is $\cos^{-1}\left(\frac{2}{3}\right)$. Find the ratio $BA : AC$ such that C, which is on the line AB, is nearest to O.	[4]
	The position vector $\mathbf{v}_k$ is such that $\mathbf{v}_k = \mathbf{a} + k(\mathbf{b}-\mathbf{a})$ where k is a positive integer. If $\overrightarrow{OS} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \ldots + \mathbf{v}_n$,	
	(iii) find $\overrightarrow{OS}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and n. Simplify your answer.	[2]
	(iv) If $\lambda = 2$ and $\mu = 3$, find the area of triangle OCS in the form $k\lvert\mathbf{a}\times\mathbf{b}\rvert$, where k is in terms of n.	[3]
	Solution	
	(i) $\mathbf{a} = \frac{\mu\mathbf{b} + \lambda\mathbf{c}}{\lambda+\mu}$	
	$\mathbf{c} = \frac{(\lambda+\mu)\mathbf{a} - \mu\mathbf{b}}{\lambda}$	
	(ii) $(\mathbf{b}-\mathbf{a})\cdot\mathbf{c} = 0$ ⟶ use scalar pdt $\overrightarrow{BA}\cdot\overrightarrow{OC} = 0$	
	$(\mathbf{b}-\mathbf{a})\cdot\left[\frac{(\lambda+\mu)}{\lambda}\mathbf{a} - \frac{\mu}{\lambda}\mathbf{b}\right] = 0$	
	$\mathbf{a}\cdot\mathbf{b}\left(\frac{\lambda+\mu}{\lambda} + \frac{\mu}{\lambda}\right) - \frac{\lambda+\mu}{\lambda}\lvert\mathbf{a}\rvert^2 - \frac{\mu}{\lambda}\lvert\mathbf{b}\rvert^2 = 0$	
	$\frac{2}{3}\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert(\lambda+2\mu) - (\lambda+\mu)\lvert\mathbf{a}\rvert^2 - \mu\lvert\mathbf{b}\rvert^2 = 0$	
	$2(\lambda+2\mu) - (\lambda+\mu) - 9\mu = 0$	
	$\lambda = 6\mu$	
	$BA : AC = 6:1$	
	Alternative Method	
	$BA = \sqrt{OA^2 + OB^2 - 2(OA)(OB)\cos\measuredangle AOB} = \sqrt{1^2 + 3^2 - 2(1)(3)\left(\frac{2}{3}\right)} = \sqrt{6}$	
	$\mathbf{a}\cdot\mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\cos\measuredangle AOB = (1)(3)\left(\frac{2}{3}\right) = 2$	

Commented [KML19]: Presentation of Answer
Simplify the answer.



Commented [KML20]: Careless Mistake
Some students use b . c = 0 which is wrong.
Presentation of Answer
A number of students used
$\overrightarrow{OC}\cdot\overrightarrow{BC} = 0$
$\left[\frac{(\lambda+\mu)}{\lambda}\mathbf{a} - \frac{\mu}{\lambda}\mathbf{b}\right].(\mathbf{c}-\mathbf{b}) = 0$
$\left[\frac{(\lambda+\mu)}{\lambda}\mathbf{a} - \frac{\mu}{\lambda}\mathbf{b}\right].\left(\frac{(\lambda+\mu)}{\lambda}\mathbf{a} - \frac{\mu}{\lambda}\mathbf{b} - \mathbf{b}\right) = 0$
which makes the manipulation more complicated.



[Turn Over

	$AC = \frac{\lvert\mathbf{a}\cdot(\mathbf{a}-\mathbf{b})\rvert}{\lvert\mathbf{a}-\mathbf{b}\rvert} = \frac{\lvert\mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b}\rvert}{\lvert\mathbf{a}-\mathbf{b}\rvert} = \frac{\lvert\lvert\mathbf{a}\rvert^2 - \mathbf{a}\cdot\mathbf{b}\rvert}{BA} = \frac{\lvert 1^2 - 2\rvert}{\sqrt{6}} = \frac{1}{\sqrt{6}}$	
	$\frac{BA}{AC} = \frac{\sqrt{6}}{\left(\frac{1}{\sqrt{6}}\right)} = 6$	
	$\therefore BA : AC = 6:1$	
	(iii) $\overrightarrow{OS} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \ldots + \mathbf{v}_n$	
	$= [\mathbf{a} + (\mathbf{b}-\mathbf{a})] + [\mathbf{a} + 2(\mathbf{b}-\mathbf{a})] + [\mathbf{a} + 3(\mathbf{b}-\mathbf{a})] + \ldots + [\mathbf{a} + n(\mathbf{b}-\mathbf{a})]$	
	$= n\mathbf{a} + \frac{n(n+1)}{2}(\mathbf{b}-\mathbf{a})$	
	$= -\frac{n(n-1)}{2}\mathbf{a} + \frac{n(n+1)}{2}\mathbf{b}$ $\overrightarrow{OC} \times \overrightarrow{OS}$	
	(iv) area of $\Delta OCS = \frac{1}{2}\left\lvert\left(\frac{5\mathbf{a}}{2} - \frac{3\mathbf{b}}{2}\right) \times \left[-\frac{n(n-1)}{2}\mathbf{a} + \frac{n(n+1)}{2}\mathbf{b}\right]\right\rvert$	
	$= \frac{1}{2}\left\lvert\frac{5n(n+1)}{4}\mathbf{a}\times\mathbf{b} + \frac{3n(n-1)}{4}\mathbf{b}\times\mathbf{a}\right\rvert$ $(\because \mathbf{a}\times\mathbf{a} = 0$ and $\mathbf{b}\times\mathbf{b} = 0)$	
	$= \frac{1}{2}\left\lvert\frac{5n(n+1)}{4}\mathbf{a}\times\mathbf{b} - \frac{3n(n-1)}{4}\mathbf{a}\times\mathbf{b}\right\rvert$ $(\because \mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a})$	
	$= \frac{n(n+4)}{4}\lvert\mathbf{a}\times\mathbf{b}\rvert$	
7	(a) The function f is such that $f : x \mapsto -2x^2 + 4x + 1, \quad x \in \mathbb{R}, x \le a$. greatest value $= 1$	
	State the greatest value of a such that f^{-1} exists. For this value of a, find f^{-1} in a similar form.	[4]
	(b) The function g is a strictly decreasing and continuous function such that $g(x) = -g(-x)$ for $x \in \mathbb{R}$. The coordinates of certain points on the curve of $y = g(x)$ are as follows: (table below)	
	(i) State the value of $g^{-1}(-6)$.	[1]
	Another function h is defined by $h : x \mapsto 4\sin 2x$ for $0 \le x \le \frac{\pi}{2}$.	
	(ii) Find the exact range of the composite function gh. Find the range of values of x such that the composite function gh satisfies the inequality $\lvert g^{-1}h(x) + 3\rvert \le 2$. Leave your answer in exact form.	[4]

x	−6	−5.7	−5	−4	−3.2	−3	−1	0	2	4.3
y	7	$4\sqrt{3}$	6	$3\sqrt{3}$	4.3	4	$2\sqrt{3}$	0	−3.7	−5.5

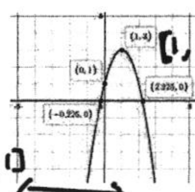
Commented [KML21]: Careless Mistake
1) A number of students didn't know the formula for sum of AP.
2) Quite a number of students used the formula $\frac{n}{2}[2\mathbf{v}_1 + (n-1)(\mathbf{b}-\mathbf{a})]$ but put $\mathbf{v}_1 = \mathbf{a}$ instead of b.

Commented [KML22]: Presentation of Answer
A large number of students didn't put in the appropriate brackets for the 2 vectors.

Commented [KML23]: Misconception
Some students remove the modulus sign and wrote
$= \frac{1}{2}\left(\frac{5n(n+1)}{4}\lvert\mathbf{a}\times\mathbf{b}\rvert + \frac{3n(n-1)}{4}\lvert\mathbf{b}\times\mathbf{a}\rvert\right)$
$= \frac{1}{2}\left(\frac{5n(n+1)}{4}\lvert\mathbf{a}\times\mathbf{b}\rvert - \frac{3n(n-1)}{4}\lvert\mathbf{a}\times\mathbf{b}\rvert\right)$
which is wrong as
$\lvert\mathbf{b}\times\mathbf{a}\rvert = \lvert\mathbf{a}\times\mathbf{b}\rvert$ but $\mathbf{b}\times\mathbf{a} = -(\mathbf{a}\times\mathbf{b})$.

Commented [KML24]: Misconception
Students shouldn't remove the modulus sign and write expression as
$\frac{1}{2}\left(\frac{5n(n+1)}{4} - \frac{3n(n-1)}{4}\right)\lvert\mathbf{a}\times\mathbf{b}\rvert$
as it is not clear whether the expression in n is a positive value. They should put in the modulus sign, i.e. $\frac{1}{2}\left\lvert\frac{5n(n+1)}{4} - \frac{3n(n-1)}{4}\right\rvert\lvert\mathbf{a}\times\mathbf{b}\rvert$ and simplify the expression in n until it is certain that the expression is positive before removing the modulus sign.

Commented [LT25]: Question Reading
Did not express the answer in similar form.



$[1, \infty)$

$(-\infty, 1]$

$(-\infty, 3]$, $x \le 3$

$R_h \subseteq D_g$

$[0, 4]$ ⟶ x values of $g(x)$

$g(0) = 0$

BP

BP-55

Domain of g^{-1} ← Range of g

$g^{-1}(-6)$ ← $g(-6) = -g(6) = -[-5] = 5$

similar form

$g(4) = -g(-4) = -3\sqrt{3}$

$R_{gh} = [-3\sqrt{3}, 0]$

9

(c) The function k is defined by

$$k : n \mapsto \begin{cases} \sqrt{n} + \frac{n}{2}, & \text{for odd positive integer } n \\ n, & \text{for even positive integer } n. \end{cases}$$

$\sqrt{1} + \frac{1}{2} = \frac{3}{2}$ = Rational no?

Explain clearly whether the composite function k^2 exists. [1]

Solution

$R_k = \frac{3}{2}$

$R_k \subset D_k$

(a) Greatest value of a is 1.

$y = -2(x-1)^2 + 3$

$x = 1 \pm \sqrt{\frac{3-y}{2}}$

$\therefore x = 1 - \sqrt{\frac{3-y}{2}} \quad (\because x \le 1)$ ✓

So $f^{-1} : x \mapsto 1 - \sqrt{\frac{3-x}{2}}, \quad x \in \mathbb{R}, x \le 3.$ R_f

(bi) $g^{-1}(-6) = 5$

OR

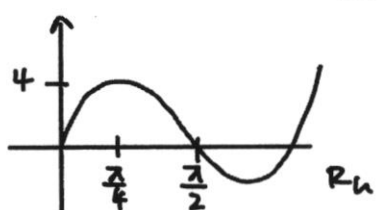
Let $g^{-1}(-6) = k$

$-6 = g(k)$

$6 = -g(k)$

$6 = g(-k)$

Hence $k = -5$ so $g^{-1}(-6) = 5$



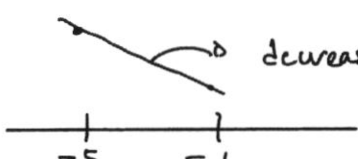
$R_h = [0, 4]$

(ii) $R_h = [0, 4]$

$g(4) = -g(-4) = -3\sqrt{3}$

$R_{gh} = [-3\sqrt{3}, 0]$

[Turn Over



Commented [LT26]: Misconception
Many thought that $R_k = [1, \infty)$ and $D_k = [1, \infty)$. One needs to note that this is not a continuous curve. It actually consists of infinitely many standalone points plotted on the Cartesian Plane.

Commented [LT27]: Presentation of Answer
A number of students missed out the word greatest.

Commented [LT28]: Careless Mistake
Some students did not complete the square correctly.

Commented [NCY29]: Misconception
When taking square root on both sides, it is expected that there are 2 possibilities, ±

Commented [LT30]: Presentation of Answer
Did not provide a reason for the selection.

Commented [LT31]: Question Reading
Did not write the answer in similar form.

Incorrect Presentation

$f^{-1}(x) = 1 - \sqrt{\frac{3-x}{2}}, \quad x \in (-\infty, 3].$

$f^{-1}(x) = 1 - \sqrt{\frac{3-x}{2}}, \quad D_{f^{-1}} = (-\infty, 3].$

Commented [LT32]: Question Reading/ Understanding of Question
A number of students gave the answer as 7.

Commented [LT33]: Misconception
$R_{gh} \ne R_g$ and $D_g \ne R_h$

$g(-5) < g(g^{-1}(h(x))) < g(-1)$

$g(-5) > h(x) > g(-1)$

$6 > h(x) > 2\sqrt{3}$

10

$-2 < g^{-1}h(x) + 3 < 2$

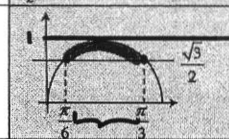
$-5 < g^{-1}h(x) < -1$

$g(-5) > h(x) > g(-1)$ Since g is a decreasing function

$6 > h(x) > 2\sqrt{3}$

$\frac{\sqrt{3}}{2} < \sin 2x < \frac{3}{2}$

$\frac{\sqrt{3}}{2} < \sin 2x \le 1$ Since $-1 \le \sin 2x \le 1$.



$\frac{\pi}{6} < x < \frac{\pi}{3}$

(c) Since $k(1) = \sqrt{1} + \frac{1}{2} = \frac{3}{2}$ is in R_k but D_k is the set of positive integers, so $R_k \not\subset D_k$. Hence k^2 does not exist.

OR

Since the range of k has irrational numbers but D_k is the set of positive integers. So $R_k \not\subset D_k$. Hence k^2 does not exist.

8 **Do not use a calculator in answering this question.**

(a)(i) Solve the equation $w^2 = 3 - 4i$, giving your answers in cartesian form $a + ib$. [4]

(ii) Hence find the roots of the equation $z^2 - 4iz + 4i - 7 = 0$, giving your answers in cartesian form $p + qi$. [2]

Commented [LT34]: Misconception
$g(-2) < h(x) + g(3) < g(2)$

Commented [LT35]: Misconception
$-5 < g^{-1}h(x) < -1$ does not imply
$-5 < g^{-1}h(x)$ or $g^{-1}h(x) < -1$. It should be
$-5 < g^{-1}h(x)$ AND $g^{-1}h(x) < -1$.

Commented [LT36]: Misconception
A number forgot to switch the sign of the inequalities as function g is a decreasing function.

Commented [LT37]: Misconceptions
A number tried to obtain the answer by directly perform inverse sine throughout and have assumed that the end values of x coincides with the end values of sin2x.

Commented [LT38]: Question Reading
A number did not take notice of this statement and so working provided is not thorough.

Commented [LT39]: Question Reading
A number convert 3-4i to exponential form whic is not ideal.

Commented [LT40]: Question Reading
Some did not know how to use the previous result to solve the current part.

	(b) For positive integer n, a complex number z is such that $\lvert z\rvert^n = \frac{1}{\sqrt{2}}$. The complex conjugate of z is z^*. State the conjugate of $1+2z^{2n}$ in terms of z^*. Hence show that [redacted] is a real number.	[5]
	Solution	
	(ai) Consider $(a+ib)^2 = 3-4i$	
	$a^2-b^2+2abi = 3-4i$	
	Comparing the real and imaginary parts	
	$a^2-b^2=3$ and $2ab=-4$	
	$\Rightarrow b = \frac{-2}{a}$	
	$a^2-\frac{4}{a^2}=3$	
	$a^4-3a^2-4=0$	
	$(a^2-4)(a^2+1)=0$	
	$a=\pm 2$ (Since a is a real number)	
	When $a=2$, $b=-1$,	
	When $a=-2$, $b=1$,	
	Hence the 2 roots are $2-i$ or $-2+i$	
	(ii) $z^2-4iz+4i-7=0$	
	Method 1	
	$z=\frac{4i\pm\sqrt{(-4i)^2-4(4i-7)}}{2}$	
	$z=\frac{4i\pm\sqrt{-16+28-16i}}{2}$	
	$z=2i\pm\sqrt{3-4i}$	
	$z=2i+(2-i)$ or $z=2i+(-2+i)$	
	$z=2+i$ or $z=-2+3i$	
	Method 2	
	$(z-2i)^2+4+4i-7=0$	
	$(z-2i)^2=3-4i$	
	$z-2i=2-i$ or $z-2i=-2+i$	
	$z=2+i$ or $z=-2+3i$	

$\frac{1+3i}{1+(2+3i)} = \frac{1+3i}{3+3i}$

$\Rightarrow a + bi$ real part

Commented [LT41]: Question Reading
Did not answer to the question.

Commented [LT42]: Question Reading
A number did not attempt the question using the conjugate of $1+2z^{2n}$.

Commented [LT43]: Many are not familiar with the properties involving complex numbers. All are strongly advised to remember them well.

Commented [LT44]: Presentation of answer
Not to be written as "Comparing coefficients"

Commented [LT45]: Careless Mistakes
Did not simplify the expression correctly.

Commented [NCY46]: Question reading
As calculator is not allowed in this question, students are expected to show all working

Commented [LT47]: Misconception
Some have forgotten that when a complex number is written in Cartesian form a+ib, **both a and b are real numbers.**

Commented [LT48]: Presentation of answer
Some did not split the values of a and b found in this way. Make sure one needs to explicitly indicate which a value is paired to which b value.

Commented [LT49]: Misconception
It is not correct write $z=2i\pm(2-i)$ or $z=2i\pm(-2+i)$.
It actually means $z=2i+(\pm\sqrt{3-4i})$
Where $\pm\sqrt{3-4i}$ are the 2 answers found in (i).

	(b) $(1+2z^{2n})^* = 1+2(z^*)^{2n}$	
	$\frac{2z^n}{1+2z^{2n}} = \frac{2z^n(1+2(z^{2n})^*)}{(1+2z^{2n})(1+2(z^{2n})^*)}$	
	$= \frac{2z^n(1+2\,[redacted])}{(1+2z^{2n})(1+2(z^{2n})^*)}$	
	$= \frac{2z^n + 4\lvert z\rvert^{2n}(z^*)^n}{(1+2z^{2n})(1+2(z^{2n})^*)}$	
	Method 1	
	$= \frac{2z^n+2(z^n)^*}{\lvert 1+2z^{2n}\rvert^2}$	
	$= \frac{4\mathrm{Re}(z^n)}{\lvert 1+2z^{2n}\rvert^2} = k,\ k\in\mathbb{R}$	
	Method 2	
	$= \frac{2z^n+2(z^n)^*}{1+4\lvert z^{2n}\rvert^2+2(z^{2n}+(z^{2n})^*)}$	
	$= \frac{4\mathrm{Re}(z^n)}{2+4\mathrm{Re}(z^{2n})} = k,\ k\in\mathbb{R}$	
9	**(a)(i)** Find $\frac{d}{dx}\left[\sin(x^3+2)\right]$.	[1]
	(ii) Find $\int 2x^5\cos(x^3+2)\,dx$.	[2]
	(b)(i) Using an algebraic approach, solve the inequality $\frac{x-16}{x^2-16}\le 1$.	[3]
	(ii) Find $\int\left(\frac{x-16}{x^2-16}-1\right)dx$.	[3]
	(iii) Hence find the exact value of $\int_{-1}^{1}\left\lvert\frac{x-16}{x^2-16}-1\right\rvert dx$ in the form of $\ln\frac{a}{b}$, where a and b are real constants.	[3]
	Solution	
	(ai) $\frac{d}{dx}\left[\sin(x^3+2)\right] = 3x^2\cos(x^3+2)$	

Rationalise

Cartesian format

$\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2}$

$z^n \cdot z^{*n}$ $\Rightarrow (z\cdot z^*)^{2n}$ $\Rightarrow \lvert z\rvert^{2n}$

$2(z^n + z^{n*}) \Rightarrow 2(2\,\mathrm{Re}(z^n))$

$\lvert z\rvert^n = \frac{1}{\sqrt{2}}$

Commented [LT50]: Presentation of Answer
Make sure that conjugate of z is indicated clearly.

Commented [TCK51]: Chain rule of differentiation is expected here and a large majority got the answer.

(i) $\cos(x^3+2)\cdot(3x^2)$

(ii) $u=\frac{2}{3}x^3$ $\frac{dv}{dx} = 3x^2\cos(x^3+2)$

$\frac{du}{dx} = 3\left(\frac{2}{3}\right)x^2 = 2x^2$ $\int\frac{dv}{dx} = \sin(x^3+2)$

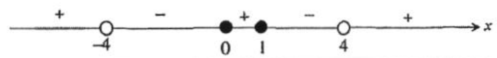
BP

BP-57

$\frac{2}{3}x^3 \sin(x^3+2) - \int$ [redacted] dx

$\Rightarrow \frac{2}{3}x^3 \sin(x^3+2) \quad -\frac{2}{3}\int 3x^2 \sin(x^3+2)\ dx$

$\frac{d}{dx}(\cos(x^3+2))$
$\Rightarrow -\sin(x^3+2) \cdot (3x^2)$
$= -3x^2 \sin(x^3+2)$

	(ii) $\int 2x^5 \cos(x^3+2)\ dx$ $\quad u = 2x^3 \quad \frac{dv}{dx} = x^2\cos(x^3+2)$ $\quad \frac{du}{dx} = 6x^2 \quad$ [redacted]	
	$= \frac{2}{3}x^3\sin(x^3+2) - \frac{2}{3}\int 3x^2\sin(x^3+2)\ dx$	
	$= \frac{2}{3}x^3\sin(x^3+2) + \frac{2}{3}\cos(x^3+2) + C$	
	(bi) $\frac{x-16}{x^2-16} \le 1$ $\Rightarrow \frac{(x-16)-(x^2-16)}{x^2-16} < 0$	
	$\frac{x-16-x^2+16}{x^2-16} \le 0$	
	$\frac{x^2-x}{x^2-16} \ge 0$ → $x = 0$ or $x = 1$ (inclusion).	
	$\frac{x(x-1)}{(x-4)(x+4)} \ge 0$ → $x \ne 4$ or -4	
		
	$x < -4$ or $0 \le x \le 1$ or $x > 4$	
	(ii) $\int \frac{x-16}{x^2-16} - 1\ dx$ $\quad \int \frac{f'(x)}{f(x)}\ dx = \frac{1}{2}\int \frac{2x}{x^2-16} - \frac{16}{x^2-16}\ dx$	
	$= \frac{1}{2}\int \frac{2x}{x^2-16}\ dx - 16\int \frac{1}{x^2-4^2}\ dx - \int 1\ dx$	
	$= \frac{1}{2}\ln\lvert x^2-16\rvert - \frac{16}{2(4)}\ln\left\lvert\frac{x-4}{x+4}\right\rvert - x + C$ $= \frac{1}{2}\ln\lvert x^2-16\rvert - 2\ln\left\lvert\frac{x-4}{x+4}\right\rvert - x + C$	
	Alternative Method $\int \frac{x-16}{x^2-16} - 1\ dx$ $= \int \frac{5/2}{x+4} - \frac{3/2}{x-4} - 1\ dx$ $= \frac{5}{2}\ln\lvert x+4\rvert - \frac{3}{2}\ln\lvert x-4\rvert - x + c$	
	(iii) When $x < -4$ or $0 \le x \le 1$ or $x > 4$, $\frac{x-16}{x^2-16} - 1 \le 0$. $\int_{-1}^{1}\left\lvert\frac{x-16}{x^2-16} - 1\right\rvert dx = \int_{-1}^{0}\frac{x-16}{x^2-16} - 1\ dx + \int_{0}^{1} -\left(\frac{x-16}{x^2-16} - 1\right) dx$	

Commented [TCK52]: Unfortunately, many of those who got (i) correct have difficulty answering this.

Commented [TCK53]: Many could not use experience of (i) to deduce the result of this integral.

Commented [CKJ54]: Common Mistake
A few students multiplied by $x^2 - 16$ throughout. This is not allowed as we are unsure whether $x^2 - 16$ is a positive or negative number.

Commented [TCK55]: Candidates are rather careless in determining the sign of the rational function for each of the intervals on the number line.

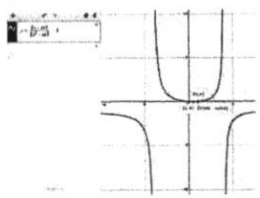
Commented [TCK56]: Many did not pay attention to the fact that x could not be ± 4.

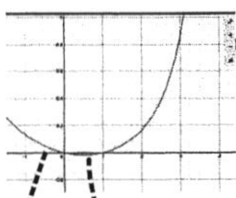
Commented [TCK57]: Many candidates left out the modulus sign especially for $\frac{x-4}{x+4}$.

Commented [CKJ58]: Many students did not know how to remove the modulus sign. Students should introduce a "minus" sign as $\frac{x-16}{x^2-16} - 1$ is negative when $0 \le x \le 1$.

Common Mistake
Some students introduce the "minus" sign to the wrong integral.

Commented [TCK59]: This proves to be difficult for a large majority of candidates. Candidates should check with their GC if their answer is correct in this case.





$+f(x) \qquad -f(x)$

	$= \int_{-1}^{0}\frac{x-16}{x^2-16} - 1\ dx + \int_{1}^{0}\frac{x-16}{x^2-16} - 1\ dx$	
	$= \left(\frac{1}{2}\ln 16 - \frac{1}{2}\ln 15 + 2\ln\left(\frac{5}{3}\right) - 1\right) + \left(\frac{1}{2}\ln 16 - \frac{1}{2}\ln 15 + 2\ln\frac{3}{5} + 1\right)$	
	$= \ln 16 - \ln 15 = \ln\left(\frac{16}{15}\right)$	
	Alternative Method $\int_{-1}^{1}\left\lvert\frac{x-16}{x^2-16} - 1\right\rvert dx =$ $\int_{-1}^{0}\frac{x-16}{x^2-16} - 1\ dx + \int_{0}^{1} -\left(\frac{x-16}{x^2-16} - 1\right) dx$ $= \left[\frac{5}{2}\ln\lvert x+4\rvert - \frac{3}{2}\ln\lvert x-4\rvert - x\right]_{-1}^{0} - \left[\frac{5}{2}\ln\lvert x+4\rvert - \frac{3}{2}\ln\lvert x-4\rvert - x\right]_{0}^{1}$ $= \left(\frac{5}{2}\ln 4 - \frac{3}{2}\ln 4\right) - \left(\frac{5}{2}\ln 3 - \frac{3}{2}\ln 5 + 1\right) - \left[\left(\frac{5}{2}\ln 5 - \frac{3}{2}\ln 3 - 1\right) - \left(\frac{5}{2}\ln 4 - \frac{3}{2}\ln 4\right)\right]$ $= \ln 4 - \frac{5}{2}\ln 3 + \frac{3}{2}\ln 5 - 1 - \frac{5}{2}\ln 5 + \frac{3}{2}\ln 3 + 1 + \ln 4$ $= 2\ln 4 - \ln 3 - \ln 5$ $= \ln 16 - \ln 15$ $= \ln\frac{16}{15}$	
10	 Container $\quad$ Cross-section of Container $\quad$ Lid	
	Crayole, a handicraft company, requires a container of uniform cross-sectional area, and with negligible thickness, to hold 200 cm³ of party slime. The cross-section of the container is a regular pentagon with sides x cm and the vertical height of the container is y cm. The sides of its lid is x cm and has a depth ky cm, where $0 < k \le 1$ (see diagram).	
	It is known that the area of a regular pentagon with sides x cm is given by $\frac{ax^2}{4}$, where a is a positive fixed constant.	
	(i) Use differentiation to find, in terms of a and k, the exact value of x which gives a minimum total external surface area of the container and the lid.	[6]

	(ii) Find the ratio $\frac{y}{x}$ in terms of a and k in this case, simplifying your answer.	[2]
	(iii) Find the range of values of $\frac{y}{x}$ in terms of a.	[2]
	(iv) Find the exact value of k, in terms of a, if the company requires the sides of the container to be $\frac{3}{4}$ of its height. sides	[2]
	Solution	
	(i) Let total external surface area (container + lid) be S.	
	$S=\left(\frac{1}{4}ax^2+5xy\right)+\left(\frac{1}{4}ax^2+5kxy\right)$ $=\frac{ax^2}{2}+5(1+k)xy$ ---------- (1)	
	Let volume of the container be V.	
	$V=\frac{ax^2y}{4}=200$ —— Base area X height	
	$y=\frac{800}{ax^2}$ ---------- (2)	
	Substitute (2) into (1)	
	$S=\frac{ax^2}{2}+5(1+k)x\left(\frac{800}{ax^2}\right)$ $=\frac{ax^2}{2}+\frac{4000(1+k)}{ax}$	
	$\frac{dS}{dx}=ax-\frac{4000(1+k)}{ax^2}$	
	For minimum S, $\frac{dS}{dx}=0$ $ax-\frac{4000(1+k)}{ax^2}=0$ $ax=\frac{4000(1+k)}{ax^2}$	
	$x=\sqrt[3]{\frac{4000(1+k)}{a^2}}$	
	To prove S is minimum, **Method 1** (2[nd] derivative test) $\frac{d^2S}{dx^2}=a+\frac{8000(1+k)}{ax^3}$ When $x=\sqrt[3]{\frac{4000(1+k)}{a^2}}$, $\frac{d^2S}{dx^2}=a+\frac{8000(1+k)}{4000(1+k)/a}=3a>0$ (since a is positive)	

Commented [AA60]: Most of the students are able to get this result. However, a few considered the volume along with the lid.

Commented [AA61]: Although students were able to write an expression for x^3, a few did not indicate cube root when they wrote the expression for x.

$x=\frac{3}{4}y$

$\frac{4}{3}=\frac{y}{x}$

[Turn Over

	Therefore, S is minimum when $x=\sqrt[3]{\frac{4000(1+k)}{a^2}}$. **Method 2** (1[st] derivative test) $\frac{dS}{dx}=\frac{a^2x^3-4000(1+k)}{ax^2}$. Since a is positive, when $x<\sqrt[3]{\frac{4000(1+k)}{a^2}}$, $a^2x^3<4000(1+k)$ and $ax^2>0 \Rightarrow \frac{dS}{dx}<0$; when $x>\sqrt[3]{\frac{4000(1+k)}{a^2}}$, $a^2x^3>4000(1+k)$ and $ax^2>0 \Rightarrow \frac{dS}{dx}>0$. Therefore, S is minimum when $x=\sqrt[3]{\frac{4000(1+k)}{a^2}}$	
	(ii) $\frac{y}{x}=\frac{800}{ax^3}$ $y=\frac{800}{ax^2}$	
	$\frac{y}{x}=\frac{800}{a\left(\frac{4000(1+k)}{a^2}\right)}=\frac{a}{5(1+k)}$	
	(iii) Since $0<k\le 1$, $1<1+k\le 2$ $\frac{1}{2}\le\frac{1}{1+k}<1$	
	[redacted] sides	
	(iv) $\frac{y}{x}=\frac{a}{5(1+k)}=\frac{4}{3}$	
	$k=\frac{3a}{20}-1$	
11	**(a)** At the beginning of January 2010, Mr Wong's fish farm has 40 000 fish breeding in a large pond. In the period between January and December each year, the number of fish increases by 9%. At the end of December each year, 5400 fish are harvested and sold at the fish markets for \$6 per fish. No fish died or are being poached while he is in business.	
	(i) Find the number of fish just after the nth harvest, giving your answer in the form $A-B(1.09)^n$, where A and B are constants to be determined.	[3]
	(ii) Find the earliest month and year in which there will be no more fish in the pond.	[2]

Commented [AA62]: Whoever attempted sign test, failed to prove it with proper steps.

Commented [KSM63]: Some only reflected the deduction of 5400 in the next year, which result in wrong pattern development.

$x=\frac{3}{4}y$ → height

$\frac{4}{3}=\frac{y}{x}$

BP

Total gains :
(left + sold over n yrs)
\$8 on (i) + \$6(5400)(n)

Mr Wong has been thinking of retiring since January 2010. Every December just after the harvest, a nearby fish farm offers to buy over Mr Wong's business by paying \$8 per fish for his remaining stock.

(iii) If he wishes to retire by selling off his fish farm business after the nth harvest, find the value of n where the total gains is the maximum. [3]

(b) White-tailed deer can be found in forests of Southern Canada. A wildlife biologist is investigating the change of a population of white-tailed deer of size x (thousands) at time t (years). He observed that on average, their birth rate is 4000 per year and their death rate is proportional to the square of the population present.

(i) When the population was 5000, the rate of increase of the population at that instant was 1750 per year. Assuming that x and t are continuous variables, form a differential equation relating x and t. [2]

(ii) There were 2000 deer in the forest initially. Solve this differential equation to obtain x as an exact function of t. [4]

$x = 5$

$\frac{dx}{dt} = 1.75$

x (thousands), t (yrs)

$T_n = ar^{n-1}$

Solution

(ai)

n	Number of fish remaining after nth harvest
1	$40000\,(1.09) - 5400$
2	$= [40000\,(1.09) - 5400]\,(1.09) - 5400$ $= 40000\,(1.09)^2 - 5400\,(1 + 1.09)$
3	$= [40000\,(1.09)^2 - 5400\,(1.09 + 1)]\,(1.09) - 5400$ $= 40000\,(1.09)^3 - 5400\,(1 + 1.09 + 1.09^2)$
	…

Number of fish after the nth harvest

$= 40000\,(1.09)^n - 5400\,(1 + 1.09 + 1.09^2 + \dots + 1.09^{n-2} +$ [redacted]$)$

G.P. with common ratio 1.09 and 1st term 1

$= 40000\,(1.09)^n - 5400\left(\frac{1.09^n - 1}{1.09 - 1}\right)$

$= 40000(1.09)^n - 60000(1.09^n - 1)$

$= 60000 - 20000(1.09)^n$, where $A = 60000$, $B = 20000$

(ii) When there is no fish,

$60000 - 20000(1.09)^n \le 0$

Method 1

$(1.09)^n = 3$

$n = \frac{\ln 3}{\ln(1.09)} = 12.7$ years

[Turn Over

$-20\,000\,(1.09)^n \le -60\,000$

$(1.09)^n \geq 3$

$n \ln(1.09) \geq \ln 3$

$n \geq \frac{\ln 3}{\ln(1.09)}$

≥ 12.7

$n = 13$ yrs

Jan 2010 – Dec 2022

13 yrs.

Method 2

n	$60000 - 20000\,(1.09)^n$
11	8391.5
12	3746.7
13	−1316

After 13 years there will be no fish, in Dec 2022. (as harvest occurs at end of year)

Commented [KSM64]: 1)Majority forgot to include 2010 as the first year, hence many give answer as Dec 2023;
2)Many also gave answer as Aug/Sept 2022. Note that at end of year 12 (2021), there's still 3746 fish. In year 13 before harvest, there's 4083 fish. So only after 13th harvest at end of Dec, then there is no fish.

(iii) n years after harvest,

Total income (I) = \$8 × (no of remaining fish) + \$6 × (no. of fish sold)

$I = 8\,[60000 - 20000\,(1.09)^n] + 6(5400)\,n$

From GC,

n	$I = 8\,[60000 - 20000\,(1.09)^n] + 6(5400)\,n$
9	424097.08
10	425221.81
11	423531.78

no. of years to maximise total income = 10

(bi) $\frac{dx}{dt} = 4 - kx^2$

Commented [KSM65]: Many did not divide the population figures by 1000

~~When $x = 5$, $\frac{dx}{dt} = 1.75$~~

x: population.

$\frac{dx}{dt} = 4 - kx^2$, $k > 0$

$1.75 = 4 - 25k$

$k = 0.09$

$\therefore \frac{dx}{dt} = 4 - 0.09x^2$

(ii) $\frac{dx}{dt} = \frac{9}{100}\left(\frac{400}{9} - x^2\right)$

$\int \frac{1}{\left(\frac{20}{3}\right)^2 - x^2}\,dx = \int \frac{9}{100}\,dt$

MF 26

$\frac{1}{2\left(\frac{20}{3}\right)} \ln\left|\frac{\frac{20}{3} + x}{\frac{20}{3} - x}\right| = \frac{9}{100}t + c$

Commented [KSM66]: Many did not put modulus. Here we are not sure if $\frac{20}{3} - x$ is positive or not

$\ln\left|\frac{20 + 3x}{20 - 3x}\right| = \frac{40}{3}\left(\frac{9}{100}t\right) + \frac{40}{3}c$

$\frac{20 + 3x}{20 - 3x} = \pm e^{\frac{40}{3}c} e^{\frac{6}{5}t}$

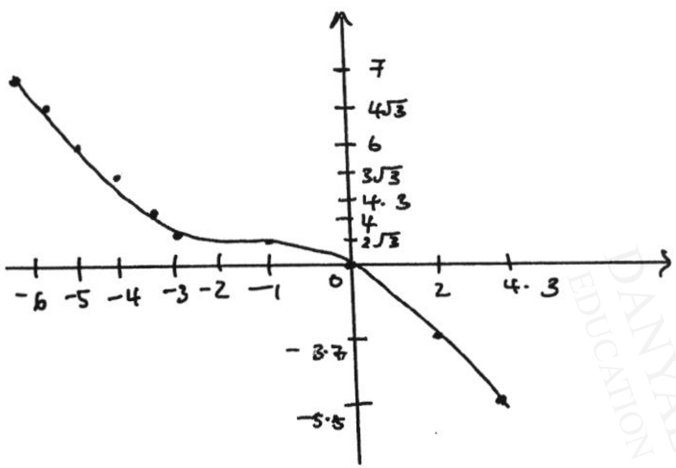
$\frac{20 + 3x}{20 - 3x} = Ae^{\frac{6}{5}t}$ where [redacted]

When $t = 0$, $x = 2$,

	$A=\frac{13}{7}$	
	$\therefore\ 20+3x=\frac{13}{7}e^{\frac{6}{5}t}(20-3x)$	
	$3x\left(1+\frac{13}{7}e^{\frac{6}{5}t}\right)=20\left(\frac{13}{7}e^{\frac{6}{5}t}-1\right)$	
	$x=\frac{20\left(13e^{\frac{6}{5}t}-7\right)}{3\left(7+13e^{\frac{6}{5}t}\right)}$	

End of Paper

x	−6	−5.7	−5	−4	−3.2	−3	−1	0	2	4.3
y	7	$4\sqrt{3}$	6	$3\sqrt{3}$	4.3	4	$2\sqrt{3}$	0	−3.7	−5.5



ANDERSON SERANGOON JUNIOR COLLEGE

MATHEMATICS **9758**

H2 Mathematics Paper 2 (100 marks) **18 Sept 2020**

3 hours

Additional Material(s): List of Formulae (MF 26)

CANDIDATE NAME

CLASS /

READ THESE INSTRUCTIONS FIRST

Write your name and class in the boxes above.
Please write clearly and use capital letters.
Write in dark blue or black pen. HB pencil may be used for graphs and diagrams only.
Do not use staples, paper clips, glue or correction fluid.

Answer **all** the questions and write your answers in this booklet.
Do not tear out any part of this booklet.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
You are expected to use an approved graphing calculator.
Where unsupported answers from a graphing calculator are not allowed in a question, you are required to present the mathematical steps using mathematical notations and not calculator commands.

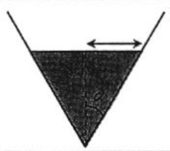
All work must be handed in at the end of the examination. If you have used any additional paper, please insert them inside this booklet.
The number of marks is given in brackets [ ] at the end of each question or part question.

Question number	Marks
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

This document consists of 18 printed pages and 2 blank pages.

[Turn Over

Section A: Pure Mathematics [40 marks]

1	The complex number z and w are given by $z = 2e^{i\theta}$, where $\frac{3\pi}{4} < \theta \le \pi$, and $w = 1 + \sqrt{3}i$. Express $w - z$ in exponential form $re^{i\alpha}$, where both r and α are in terms of θ and $-\pi < \alpha \le \pi$.	[3]
	Solution	
	Express as $w = 2e^{i\frac{\pi}{3}}$.	
	$w - z = 2e^{i\frac{\pi}{3}} - 2e^{i\theta}$	
	$= 2e^{i\left(\frac{\pi+3\theta}{6}\right)}\left(e^{i\left(\frac{\pi-3\theta}{6}\right)} - e^{-i\left(\frac{\pi-3\theta}{6}\right)}\right)$	
	$= 2e^{i\left(\frac{3\theta+\pi}{6}\right)} \times 2i \times \underbrace{\sin\left(\frac{\pi-3\theta}{6}\right)}_{\text{negative}}$ [$\frac{\pi-3\theta}{6}$ is in the 4th quadrant]	
	$= 2e^{i\left(\frac{3\theta+\pi}{6}\right)} \times 2i \times (-1)\underbrace{\sin\left(\frac{3\theta-\pi}{6}\right)}_{\text{positive}}$ [since $\sin(-x) = -\sin x$]	
	$= 4\sin\left(\frac{3\theta-\pi}{6}\right)e^{i\left(\frac{3\theta+\pi}{6}\right)}e^{i\left(-\frac{\pi}{2}\right)}$ [since $-i = e^{i\left(-\frac{\pi}{2}\right)}$]	
	$= 4\sin\left(\frac{\theta}{2} - \frac{\pi}{6}\right)e^{i\left(\frac{\theta}{2} - \frac{\pi}{3}\right)}$	
2	(a) A curve is defined by the parametric equations $x = \frac{1}{t}$, $y = t^2$, for $0 < t < 1$. Show that the equation of the normal to the curve at the point $P\left(\frac{1}{p}, p^2\right)$ is $2p^4y - px = 2p^6 - 1$. Hence, using an algebraic method, show that the normal at P cuts the curve exactly once.	[6]
	(b) 	

Commented [LMH1]: Misconception
A worrying observation is that some students erroneously think that
|w + z| = |w| + |z|,
arg(w + z) = arg(w) + arg(z).
These are false! For these students, do make it a point to clarify with your tutors.

Commented [LMH2]: Things to note
Taking out "the average of the arguments" will help surface addition/subtraction of complex conjugates. And we know that this can then be simplified.

Commented [LMH3]: Things to note
For exponential form $re^{i\theta}$, the modulus r must be a positive real number.

$\frac{\pi}{6} - \frac{\theta}{2}$; $\Rightarrow \frac{3\pi}{4} < \theta \le \pi$; $\frac{3\pi}{8} < \frac{\theta}{2} \le \frac{\pi}{2}$; $\Rightarrow \frac{\pi}{6} - \frac{3\pi}{8}$ and $\frac{\pi}{6} - \frac{\pi}{2}$; $\Rightarrow -\frac{5\pi}{24}$ and $-\frac{\pi}{3} = -\frac{8\pi}{24}$

	A cone of semi-vertical angle α, where $\tan\alpha = \frac{1}{3}$, is held with its axis vertical and vertex downwards (see diagram). At the beginning of an experiment, it is filled with 300 cm³ of water. Water runs out of a small hole at the vertex at a constant rate of 0.2 cm³ per second. At time t minutes after the start, the radius of the water surface is r cm (see diagram). Find the rate at which the depth of the water is decreasing 5 minutes after the start of the experiment. [The volume of a cone of base radius r and height h is given by $V = \frac{1}{3}\pi r^2 h$.]	[5]
	Solution	
	(a) $x = \frac{1}{t}$ $\quad y = t^2$; $\frac{dx}{dt} = -\frac{1}{t^2}$ $\quad \frac{dy}{dt} = 2t$	
	$\frac{dy}{dx} = \frac{2t}{-\frac{1}{t^2}} = -2t^3$	
	Equation of normal at point P: $y - p^2 = \frac{1}{2p^3}\left(x - \frac{1}{p}\right)$	
	$2p^4y - 2p^6 = px - 1$; $2p^4y - px = 2p^6 - 1$	
	To check if the normal cuts the curve again, $2p^4(t^2) - p\left(\frac{1}{t}\right) = 2p^6 - 1$	
	$2p^4t^3 - p + (1 - 2p^6)t = 0$	
	$(t - p)(2p^4t^2 + At + 1) = 0$	
	Comparing coefficient of t: $1 - Ap = 1 - 2p^6$; $A = 2p^5$; $\therefore (t - p)(2p^4t^2 + 2p^5t + 1) = 0$; $t = p$ or $2p^4t^2 + 2p^5t + 1 = 0 \cdots\cdots(1)$	
	For $2p^4t^2 + 2p^5t + 1 = 0$, Discriminant $= (2p^5)^2 - 4(2p^4)(1)$	
	$= 4p^{10} - 8p^4$; $= 4p^4(p^6 - 2) < 0$ since $0 < p < 1$ (Need to state the reason)	

[Turn Over

Commented [SH4]: Things to note:
This is a crucial step for students to check if there is any other solution if the normal and the curve were to intersect.

Commented [SH5]: Things to Note:
Since Point P can be the only solution, $t = p$ is the only root. Therefore, expressing $(t - p)(2p^4t^2 + At + 1)$ is important.

Commented [SH6]: Presentation
Algebraic Method is only acceptable.
Students should make it a point to show that, Discriminant $4p^{10} - 8p^4 < 0$ and state the reason why this is so. Since it's a show question.

$t = p$ $\left(\frac{1}{p}, p^2\right)$; cubic function; quadratic $ax^2 + bx + c$; since $a = 2p^4$, $c = 1$ (by observation) then $b = A \times t$

Long Division.
$$(t - p)\overline{)2p^4t^3 - p + (1 - 2p^6)t} = 2p^4t^2 + 2p^5t + 1$$
$-(2p^4t^3 - 2p^5t^2)$
$2p^5t^2 - p + t - 2p^6t$
$-(2p^5t^2 \quad - 2p^6t)$

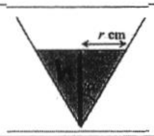
Hence (1) has no real solution.
Since there is only 1 real solution for t, so the normal at P cuts the curve only once.

(b) Let V be the volume of the water in the cone and h be the depth of water in the cone. Then,

$V=\frac{1}{3}\pi r^2 h$

$\tan\alpha=\frac{r}{h}=\frac{1}{3}$

$r=\frac{1}{3}h$



$V=\frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h=\frac{1}{27}\pi h^3$

Initial volume = 300 cm³
Decrease in volume after 5 min $=0.2\times60\times5$
$=60$ cm³
Volume of the water in the cone = 240 cm³

$240=\frac{1}{27}\pi h^3$

$h^3=\frac{6480}{\pi}$

$h=\sqrt[3]{\frac{6480}{\pi}}$

$\frac{dV}{dh}=\frac{1}{9}\pi h^2$

$\frac{dV}{dh}=\frac{1}{9}\pi h^2$
$\frac{dV}{dt}=\frac{dV}{dh}\times\frac{dh}{dt}$
$-0.2=\frac{1}{9}\pi\left(\frac{6480}{\pi}\right)^{\frac{2}{3}}\frac{dh}{dt}$

$\frac{dh}{dt}=-0.00354$ cm/s
The height is decreasing at a rate of 0.00354 cm/s.

3 A drone carrying a bomb departs from a point $A(-1,1,-3)$. It flies in a direction of $-a\mathbf{i}+\mathbf{j}+2a\mathbf{k}$, where $a>0$, across a lake to a target location. It is given that the surface of the lake is part of a plane with equation $x-z=2$.

(i) Determine the value of a if the path of the drone makes an angle of $\frac{\pi}{3}$ radians with the surface of the lake. [3]

It is given that $a=2$.

(ii) A missile is launched to intercept the drone. It moves in a path with equation $\frac{x-10}{2}=z,\ y=m$, where m is a real constant. Given that the missile is successful in intercepting the drone, find the point of interception. [3]

(iii) At the point of interception, a piece of the drone falls perpendicular to the lake and meets the surface of the lake at N. Find the position vector of N. [3]

(iv) Denoting the line that contains the drone's path as L_1 and L_1' being the reflection of L_1 in the surface of the lake, find a vector equation of L_1'. [3]

Solution

(i)
$$\sin\frac{\pi}{3}=\frac{\begin{pmatrix}-a\\1\\2a\end{pmatrix}\cdot\begin{pmatrix}1\\0\\-1\end{pmatrix}}{\sqrt{5a^2+1}\sqrt{2}}$$

$(6a)^2=6(5a^2+1)$

$a=1$ or $a=-1$ (rejected since $a>0$)

(ii) Given that $a=2$.

$l_{\text{drone}}:\ \mathbf{r}=\begin{pmatrix}-1\\1\\-3\end{pmatrix}+\lambda\begin{pmatrix}-2\\1\\4\end{pmatrix},\lambda\in\mathbb{R}$

$l_{\text{missile}}:\ \mathbf{r}=\begin{pmatrix}10\\m\\0\end{pmatrix}+\mu\begin{pmatrix}2\\0\\1\end{pmatrix},\mu\in\mathbb{R}$

[redacted] $1+\lambda$ = m, [redacted] 4λ = μ

$-2\mu-2\lambda=11$
$-\mu+4\lambda=3$

$\lambda=-\frac{1}{2},\ \mu=-5$

$m=\frac{1}{2}$

Coordinates of point of intersection are $\left(0,\frac{1}{2},-5\right)$.

[Turn Over

Commented [KSM7]: Need to state the implication of Disc < 0 to fully answer why there's only 1 real root.

Commented [KSM8]: Things to note: Some attempted to use V in terms of r to solve, but often were unable to continue to get $\frac{dh}{dt}$, so lost the marks for linking to the depth. There were very few successful attempts though, who see that $\frac{dh}{dt}=3\frac{dr}{dt}$.

Commented [SH9]: Careless Mistake: Many students have used $\frac{dV}{dt}=0.2$ instead of $\frac{dV}{dt}=-0.2$

Commented [SH10]: Presentation
1. $\frac{dh}{dt}=-0.00354$ cm/s
2. Rate of decrease = 0.00354 cm/s. mean the same but said differently.

Commented [NCY11]: Question reading Students didn't realise this is the direction vector of the path and not a position vector of a point.

Commented [NCY12]: Question reading Students didn't realise this given condition is needed to reject the value of a

Commented [LT13]: Question Reading A number did not realized that this is the angle between a line and a plane.

Commented [LT14]: Misconception A number of students could not convert this equation accurately in vector equation form.

Commented [LT15]: Misconception A number of students failed to put a modulus on the right hand side of the equation. This is because $\sin\frac{\pi}{3}$ is positive whereas the dot product on the right hand side may give rise to a negative outcome if not careful.

Commented [LT16]: Presentation of Answe Some did not explain why the value of a is 1 and not -1.

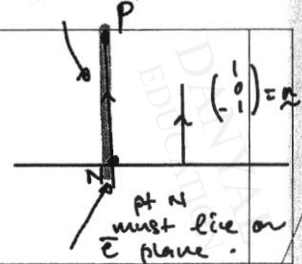
Commented [NCY17]: Misconception Many are unable to convert to the vector equation accurately

Commented [NCY18]: Question reading As question ask for a point, the answer must be presented in a coordinate form

As question didn't ask for the point in terms of m, thus it is expected the answer is a value. In any case, the value of m is not needed to solve for the point

point of inception

Equation of line



	(iii) Let the point of interception be P. $\vec{ON}$ $l_{NP}: \underset{\sim}{r} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -5 \end{pmatrix} + \alpha \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$, $\alpha \in \mathbb{R}$ for same Since N also lies on the plane $\begin{pmatrix} \alpha \\ \frac{1}{2} \\ -5-\alpha \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2$ for some $\alpha \in \mathbb{R}$	
	$2\alpha = -3$	
	$\alpha = -\frac{3}{2}$ $\quad \underset{\sim}{r} \cdot \underset{\sim}{n} = \underset{\sim}{a} \cdot \underset{\sim}{n} = 2$	
	$\vec{ON} = \frac{1}{2}\begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix}$	
	OR $\vec{NP} = \left[\begin{pmatrix} 1 \\ -\frac{1}{2} \\ -2 \end{pmatrix} \cdot \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{2}} \right] \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{2}}$	
	$= \frac{1}{2}\begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix}$	
	$\vec{ON} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ -5 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 3 \\ 0 \\ -3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix}$	
	(iv) Let P' be the reflected point of P in the lake. Using ratio theorem, $\vec{ON} = \frac{\vec{OP} + \vec{OP'}}{2}$ $\Rightarrow \vec{OP'} = 2\vec{ON} - \vec{OP} = \begin{pmatrix} -3 \\ 1 \\ -7 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ -10 \end{pmatrix} = \begin{pmatrix} -3 \\ \frac{1}{2} \\ -2 \end{pmatrix}$	

Commented [LT19]: Misconception
Wrote $\vec{PN} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2$ or $\vec{AN} \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = 2$

Commented [LT20]: Misconception
Most who did using the projection vector approach often ends up getting the wrong answer because of a wrong formula used or a wrong direction of the vector quoted. It is highly recommended **NOT TO USE the projection vector method.**

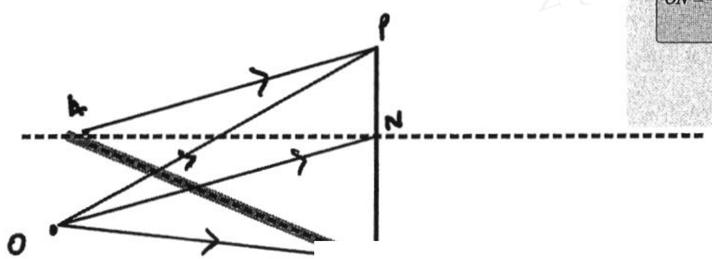
Incorrect to write is as
$\vec{PN} = \left| \begin{pmatrix} 1 \\ -\frac{1}{2} \\ -2 \end{pmatrix} \cdot \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{2}} \right| \frac{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}}{\sqrt{2}}$ because there is no modulus in the formula when using the projection vector approach.

Commented [NCY21]: Misconception
Midpoint theorem needed to be stated accurately

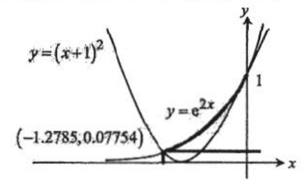
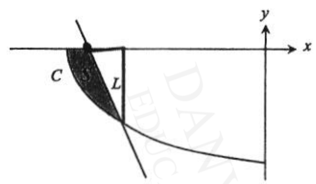
Misconception:
$\vec{ON} = \frac{\vec{AP} + \vec{AP'}}{2}$

$\vec{AN} = \frac{d_L + d_{L'}}{2}$

$\vec{ON} = \frac{\vec{OA} + \vec{OA'}}{2}$



P'

	The direction vector of the reflected line is $\begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} - \begin{pmatrix} -3 \\ \frac{1}{2} \\ -2 \end{pmatrix} = \begin{pmatrix} 2 \\ \frac{1}{2} \\ -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$	
	$\therefore L_1' : \mathbf{r} = \begin{pmatrix} -1 \\ 1 \\ -3 \end{pmatrix} + \beta \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix}$, $\beta \in \mathbb{R}$	
4	(a) Find $\int \left(\cot^6 2x + \cot^4 2x - \sin\frac{x}{2}\sin\frac{3x}{2} \right) dx$.	[3]
	(b) The diagram shows the graphs of $y = e^{2x}$ and $y = (x+1)^2$.	
		
	R is the finite region bounded by the two curves $y = e^{2x}$ and $y = (x+1)^2$. Find the volume of the solid formed when R is rotated through 2π radians about the y-axis, giving your answer correct to 4 decimal places.	[3]
	(c) A curve C has parametric equations $x = 2\left(\sin\frac{t}{2} - \cos\frac{t}{2}\right)$, $\quad y = \sin\frac{t}{2} + \cos\frac{t}{2}$, $\quad$ for $-\frac{3\pi}{2} \le t \le -\frac{\pi}{2}$.	
	The line L with equation $y = -2x - 5$ meets the curve C at the point $(-2, -1)$. S is the region enclosed by line L, curve C and the x-axis as shown in the diagram below.	
		
	(i) Show that the x-intercept of curve C is $-2\sqrt{2}$.	[2]
	(ii) Find the exact area of S.	[6]
	Solution	

$\pi \int x^2 \, dy$

$y = e^{2x}$
$\ln y = 2x \ln e$
$\therefore x = \frac{1}{2}(\ln y)$

$y = (x+1)^2$
$\pm\sqrt{y} = x+1$
$\therefore x = -1 \pm \sqrt{y}$

$P_1 + P_2 = \text{Total} \div 2$

Commented [NCY22]: Misconception
Direction vector of the reflected line cannot be the same as the original line when they are not even parallel

Commented [NCY23]: Presentations
Presentation of direction vector should always be in simplest form

[Turn Over

BP~64

$\int f'(x) \cdot f(x)^n dx = \frac{f(x)^{n+1}}{n+1} + C$

$\tan^2 x + 1 = \sec^2 x$

$\cot^2 x + 1 = \operatorname{cosec}^2 x$

Formula.

$\therefore \frac{d}{dx}(\tan x) = \sec^2 x$

$\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(a) $\int \left(\cot^6 2x + \cot^4 2x - \sin\frac{x}{2}\sin\frac{3x}{2} \right) dx$ → Factor formula.

$= \int \left(\cot^4 2x(\cot^2 2x + 1) + \frac{1}{2}(\cos 2x - \cos x) \right) dx$

$= \int \left(\cot^4 2x(\operatorname{cosec}^2 2x) + \frac{1}{2}(\cos 2x - \cos x) \right) dx$

$f(x) = \cot 2x$

$f'(x) = -2\operatorname{cosec}^2 2x$

$= -\frac{\cot^5 2x}{10} + \frac{\sin 2x}{4} - \frac{\sin x}{2} + c$

(b) Volume generated by region S

$= \pi\int_{0.077542}^{1} \left(\frac{1}{2}\ln y\right)^2 dy + \pi\int_0^{0.077542} \left(-1-\sqrt{y}\right)^2 dy - \pi\int_0^1 \left(-1+\sqrt{y}\right)^2 dy$

$= 0.5593$ (correct to 4 dec. place)

(ci) When $y = 0$,

$0 = \sin\frac{t}{2} + \cos\frac{t}{2}$

$\sin\frac{t}{2} = -\cos\frac{t}{2}$ → $\tan(\frac{1}{2}t)$

$\tan\frac{t}{2} = -1$ — same func. 2 along ē ratio

$t = -\frac{\pi}{2}$ since $-\frac{3\pi}{2} \le t \le -\frac{\pi}{2}$

$x = 2\left(\sin\left(-\frac{\pi}{4}\right) - \cos\left(-\frac{\pi}{4}\right)\right) = 2\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = -2\sqrt{2}$

(ii) Area $= -\int_{-2\sqrt{2}}^{-2} y_C \, dx - \frac{1}{2}\left(\frac{1}{2}\right)(1)$

$= -\int_{-\frac{\pi}{2}}^{-\pi} \left[\left(\sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)\right) 2\left(\frac{1}{2}\right)\left(\sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)\right) \right] dt - \frac{1}{4}$

$= \int_{-\pi}^{-\frac{\pi}{2}} \left(\sin\left(\frac{t}{2}\right) + \cos\left(\frac{t}{2}\right)\right)^2 dt - \frac{1}{4}$

$= \int_{-\pi}^{-\frac{\pi}{2}} \left[\sin^2\left(\frac{t}{2}\right) + \cos^2\left(\frac{t}{2}\right) + 2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right) \right] dt - \frac{1}{4}$

Commented [KML24]: Things to Note
1) Most students knew that they had to use the factor formula involving difference in 2 cosines to integrate $\sin\frac{x}{2}\sin\frac{3x}{2}$ but they made mistakes in the expression involving the 2 cosines especially with the angles or signs.
2) Most students didn't know how to integrate the terms in cot 2x even after they obtained $\cot^4 2x(\operatorname{cosec}^2 2x)$.

$\frac{d}{dx}(\cot 2x)$

$= -2\operatorname{cosec}^2 2x$

$-\frac{1}{2}\int -2\operatorname{cosec}^2 x \cdot \cot^4 2x \, dx$

Commented [KML25]: Presentation
Many didn't show clear working on how they obtained $t = -\frac{\pi}{2}$ from $y = 0$.
They straightaway wrote down $t = -\frac{\pi}{2}$ from $y = 0$.

Commented [KML26]: Presentation
As this is a "show" question, students need to show clearly the substitution to $-\frac{1}{\sqrt{2}}$ from $\sin\left(-\frac{\pi}{4}\right)$ and $\cos\left(-\frac{\pi}{4}\right)$.

Commented [KML27]: Presentation
Many didn't show the integral in cartesian form and straightaway changed it to the parametric form.
Misconception
Area must be positive.
1) Many missed out the minus sign in front of the integral involving y_C as $y_C < 0$.
2) Some put C in the integral instead of y_C.
3) Instead of finding the area of triangle straightaway, many used $\int_{-5/2}^{-2} (-2x-5)\,dx$, missing out the minus sign in front of the integral.
4) Some didn't change the limits of the integral when they changed the variables from x to t.

$= \int_{-\pi}^{-\frac{\pi}{2}} (1 + \sin t) \, dt - \frac{1}{4}$

$= \left[t - \cos t \right]_{-\pi}^{-\frac{\pi}{2}} - \frac{1}{4}$

$= -\frac{\pi}{2} - (-\pi - \cos(-\pi)) - \frac{1}{4}$

$= \frac{\pi}{2} - \frac{5}{4}$

Section B: Statistics [60 marks]

5 Elaine always receives two \$2 notes, two \$5 notes and one \$20 note from her parents as her monthly allowance. She decides to select two of these notes to contribute to Community Chest for the month of August. The total value of these notes is denoted by \$C.

(i) Determine the probability distribution of C. [3]

(ii) Find the expected amount of money that she will donate in August and the variance of C. [2]

In September, Elaine decides to contribute to Community Chest again.

(iii) Find the probability that her total amount of donations for August and September is at least \$12. [2]

Solution

(i) **Case 1:** P(\$2, \$2) $= P(C = 4) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$

Case 2: P(\$2, \$5) or P(\$5, \$2) $= P(C = 7) = 2 \times \left(\frac{2}{5} \times \frac{2}{4}\right) = \frac{2}{5}$

Case 3: P(\$5, \$5) $= P(C = 10) = \frac{2}{5} \times \frac{1}{4} = \frac{1}{10}$

Case 4: P(\$20, \$2) or P(\$2, \$20) $= P(C = 22) = 2 \times \left(\frac{1}{5} \times \frac{2}{4}\right) = \frac{1}{5}$

Case 5: P(\$20, \$5) or P(\$5, \$20) $= P(C = 25) = 2 \times \left(\frac{1}{5} \times \frac{2}{4}\right) = \frac{1}{5}$

C	4	7	10	22	25
$P(C = r)$	$\frac{1}{10}$	$\frac{2}{5}$	$\frac{1}{10}$	$\frac{1}{5}$	$\frac{1}{5}$

(ii) $E(C) = 4\left(\frac{1}{10}\right) + 7\left(\frac{2}{5}\right) + 10\left(\frac{1}{10}\right) + 22\left(\frac{1}{5}\right) + 25\left(\frac{1}{5}\right) = 13.6$

Expected amount of money = \$13.60
(Answer to 2 dp)

Variance of $C = E(C^2) - [E(C)]^2$ (with substitutions)

Commented [TCK28]: Quite a number of candidates did not read the question with care. First, this is selection of 2 notes without replacement; second, there is only one \$20 note

Commented [TCK29]: Failure to take into consideration the factor '2!' - a \$2 note followed by a \$5 note or vice-versa.
Recommendation for those who tend to forget about the number of permutations – use combination
e.g. In Case 2, probability $= \frac{{}^2C_1 \times {}^2C_1}{{}^5C_2} = \frac{2}{5}$
In Case 4, probability $= \frac{{}^1C_1 \times {}^2C_1}{{}^5C_2} = \frac{1}{5}$

Commented [TCK30]: Reminder: Always leave answer in 2 d.p. if variable is money.

$E(C^2) = 4^2\left(\frac{1}{10}\right) + 7^2\left(\frac{2}{5}\right) + 10^2\left(\frac{1}{10}\right) + 22^2\left(\frac{1}{5}\right) + 25^2\left(\frac{1}{5}\right)$

$= 1.6 + 19.6 + 10 + 96.8 + 125$

$= 253$

Var $(C) = 253 - (13.6)^2 = 68.04$

(iii) Complementary Method

Identify cases that will be less than $12 in donations :

1. $C_1 + C_2 = 2(\$4) = \$8$
2. $C_1 + C_2 = (\$4) + (\$7) = \$11$

$P(C_1 + C_2 \geq 12) = 1 - P(C_1 + C_2 < 12)$

$= 1 - \left(\left(\frac{1}{10}\right)^2 + 2\left(\frac{2}{5}\right)\left(\frac{1}{10}\right)\right) = \frac{91}{100}$

Commented [TCK31]: Candidates should bear this useful method in mind. Only a few candidates who did the direct approach of finding total donations at least $12 succeed in getting the correct answer because of the sheer number of possible cases.
Misconception: $C_1 + C_2$ is a discrete r.v., so it does make sense to attach a normal distribution to $C_1 + C_2$.
Many candidates use this wrong approach to find the probability.

6 A group of 300 students are asked whether they own any earbuds, laptops or games machines.

90 students own a pair of earbuds
177 students own a laptop
100 students own a games machine

Commented [TCK32]: Reading & Interpretation of question:
300 is the total number of students, not 90+177+100 = 367 as some candidates thought.

Events A, L and G are defined as follows:

A: a randomly chosen student owns a pair of earbuds.
L: a randomly chosen student owns a laptop.
G: a randomly chosen student owns a game machine.

It is given that events A and G are independent events. It is also given that 35 students own a laptop and a games machine and that 20 students own all the three gadgets.

(i) Find the probability that a student selected at random owns either a games machine or a pair of earbuds but not both. [2]

(ii) A student selected at random owns a games machine. Find the probability that the student owns exactly two of the gadgets. [2]

(iii) Find the greatest and least value of $P(A \cap L \cap G')$. [3]

Commented [TCK33]: Reading & Interpretation of question:
$P(A \cap G) = P(A) \times P(G) = \frac{90}{300} \times \frac{100}{300} = \frac{1}{10}$
$\Rightarrow n(A \cap G) = \frac{1}{10} \times 300 = 30$
$\Rightarrow n(A \cap G \cap L') = 30 - n(A \cap G \cap L)$
$= 30 - 20$
$= 10$
Many candidates take $A \cap G$ to mean a student owns a pair of earbuds and games machine only, which is wrong.

Solution

$P(L) = \frac{177}{300} = \frac{59}{100}$

$P(G) = \frac{100}{300} = \frac{1}{3}$

$P(A) = \frac{90}{300} = \frac{3}{10}$

$P(G \cap L) = \frac{35}{300} = \frac{7}{60}$

$P(A \cap G \cap L) = \frac{20}{300} = \frac{1}{15}$

Commented [TCK34]: Reading & Interpretation of question:
$n(L \cap G) = 35$
$n(L \cap G \cap A) = 20$
$\Rightarrow n(L \cap G \cap A') = 35 - 20 = 15$
Many candidates take $L \cap G$ to mean a student owns a laptop and games machine only, which is wrong.

Commented [TCK35]: Reading & Interpretation of question:
$n(L \cap G \cap A) = 20$

(i) Required probability $= P(G \cup A) - P(G \cap A) = \frac{100}{300} + \frac{90}{300} - 2\left(\frac{100}{300}\right)\left(\frac{90}{300}\right)$

$= \frac{13}{30}$

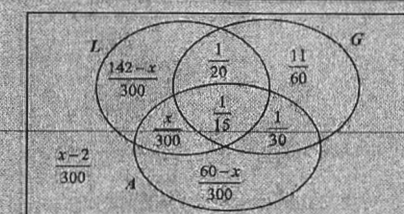
Commented [TCK36]: Misconception:
$G \cup A$ means a student selected randomly owns either a games machine or a pair of ear buds or both.
BUT question says 'a student selected at random owns either a games machine or a pair of earbuds ***but not both***'. (Note italics)

(ii) P(owns two gadgets| owns a game machine) $= \frac{15+10}{100}$ or $\frac{\frac{1}{20}+\frac{1}{30}}{\frac{1}{3}}$

$= \frac{1}{4}$

Commented [TCK37]: Reading & Interpretation of question:
Question implies a condition probability in (ii).
$Prob. = \frac{P(A \cap G \cap L') + P(L \cap G \cap A')}{P(G)}$
$= \frac{\frac{10}{300} + \frac{15}{300}}{\frac{1}{3}}$
$= \frac{\frac{1}{30} + \frac{1}{20}}{\frac{1}{3}}$
$= \frac{1}{4}$

(iii) Let $P(L \cap A \cap G') = \frac{x}{300}$



From the above, $2 \leq x \leq 60$ for all probabilities to be at least zero.

$\frac{2}{300} \leq P(A \cap L \cap G') \leq \frac{60}{300}$

$\frac{1}{150} \leq P(A \cap L \cap G') \leq \frac{1}{5}$

Commented [TCK38]: Notation:
No proper use of notation is an issue.
e.g. the right way to notate a student who owns a laptop and a pair of earbuds nut no games machine is
$L \cap A \cap G'$.
e.g. $L \cap A' \cap G'$ is the event a student owns a laptop but no earbuds and games machine
NOT L only
e.g. $L' \cap A \cap G'$ is the event a student does not own all 3 gadgets.

Commented [TCK39]: Reading & Interpretation of question:
The main problem is that candidates did not consider that there may be students who do not own all the 3 gadgets. As a result, their analysis is wrong.

BP~

7 The masses, in grams, of mangoes have the distribution $N(200, 30^2)$ and the masses, in grams, of oranges have the distribution $N(\mu, \sigma^2)$.

(i) Let A be the average mass, in grams, of a mango and two oranges. Given that $P(A < 162) = P(A > 206) = 0.06$, find μ. Hence form an equation involving σ and solve for σ. [5]

(ii) Find probability that the total mass of 3 mangoes differs from 3 times the mass of an orange by at most 136 grams. [3]

(iii) Mangoes are sold at $26 per kilogram. Find the probability that the total selling price of 6 mangoes exceeds $29.50. [2]

Solution

(i) Let X be the mass, in grams, of a randomly chosen mango.
Let Y be the mass, in grams, of a randomly chosen orange.
$X \sim N(200, 30^2)$ and $Y \sim N(\mu, \sigma^2)$

Let $A = \dfrac{X + Y_1 + Y_2}{3}$

$A \sim N\left(\dfrac{200+2\mu}{3}, \dfrac{2(\sigma^2)+30^2}{9}\right)$

$P(A < 162) = P(A > 206) = 0.06$

$\Rightarrow \dfrac{200+2\mu}{3} = \dfrac{162+206}{2}$

$\Rightarrow \dfrac{200+2\mu}{3} = 184$

$\Rightarrow \mu = 176$

$P(A > 206) = 0.06$

$P\left(Z > \dfrac{206-184}{\sqrt{\dfrac{2\sigma^2+900}{9}}}\right) = 0.06$ where $Z \sim N(0,1)$

$\dfrac{66}{\sqrt{2\sigma^2+900}} = 1.55477$

$2\sigma^2 + 900 = 1801.994907$

$\sigma = 21.237$ (5 s.f.) $= 21.2$ (3 s.f.)

(ii) Let $S = X_1 + X_2 + X_3 - 3Y$

$S \sim N(3\times200 - 3\times176,\ 3\times30^2 + 3^2\times21.237^2)$

$S \sim N(72,\ 6758.977)$

$P(|S| \le 136) = P(-136 < S < 136)$

$= 0.776$ (3 s.f.)

> **Commented [KML40]:** Question Reading
> Some took A to be the total mass instead of the average mass.

> **Commented [KML41]:** Presentation
> 1) A large number of students didn't show this expression in X and Y.
> 2) Some didn't put in the distribution for A after finding E(A) and var(A).

> **Commented [KML42]:** Mistake
> A number of students divide by 3 instead of 3^2 in finding var(A).

> **Commented [KML43]:** Misconception
> Many put $\mu = \dfrac{162+206}{2}$ instead of E(A).

> **Commented [KML44]:** Presentation and Misconception
> Many didn't show this standardization involving Z and straightaway wrote the equation below involving invnorm.
>
> In the standardization, many made the following mistakes:
> 1) $P\left(Z > \dfrac{206-\mu}{\sigma}\right)$
> 2) $P\left(Z > \dfrac{206-E(A)}{Var(A)}\right)$
> instead of $P\left(Z > \dfrac{206-E(A)}{Std\,Dev(A)}\right)$.

> **Commented [KML45]:** Misconception
> 1) Many didn't put in the modulus sign.
> 2) A common mistake was
> $P(|S| \le 136) = P(S > -136) + P(S < 136)$

(iii) Let $T = 0.026(X_1 + X_2 + X_3 + X_4 + X_5 + X_6)$

$T \sim N(0.026\times6\times200,\ 0.026^2\times6\times30^2)$

$T \sim N(31.2,\ 3.6504)$

$P(T > 29.5) = 0.813$ (3 s.f.)

> **Commented [KML46]:** Misconception
> 1) Many didn't change to grams and worked with 26 instead of 0.026.
> 2) For those who made the change, a common mistake was they divide by 1000 instead of 1000² in finding var (T), i.e.
> they put Var(T) $= \dfrac{26^2}{1000}\times6\times30^2$

8 The boss of a hair salon chain claims that the waiting time, on average, is 15 minutes. Jeremy believes that the waiting time quoted by the hair salon chain is understated. He carries out a survey to investigate this by asking the waiting time for 110 random customers. The waiting time are summarised by

$$\sum t = 1900, \text{ and } \sum (t-15)^2 = 25400.$$

(i) Find the unbiased estimate of the population mean. Leave your answers correct to 3 decimal places. Show that the unbiased estimate of the population variance is given by 227.815. [2]

(ii) Carry out a test on Jeremy's belief at the 5% significance level, stating a necessary assumption for the test. [5]

(iii) Kalai did another test to determine whether the mean waiting time differs from 15 minutes using another random sample. The waiting times of n randomly chosen customers, where n is large, were recorded, and their mean and standard deviation are 15.6 minutes and 2.5 minutes respectively. Given that the null hypothesis is rejected at the 5% level of significance, find the least value of n. [4]

Solution

(i) Unbiased estimate of population mean $= \dfrac{\sum t}{110} = \dfrac{1900}{110} = 17.273$

$\sum (t-15) = 1900 - 15(110) = 250$

Unbiased estimate of population variance $= \dfrac{1}{109}\left[\sum (t-15)^2 - \dfrac{\left[\sum (t-15)\right]^2}{110}\right]$

$\dfrac{1}{109}\left[25400 - \dfrac{250^2}{110}\right] = 227.815$

(ii) Assume that the waiting time of a customer who visit the hair salon is independent of the other customers.

Let μ be the average waiting time of a customer.
Let T be the random variable "the waiting time of a randomly chosen customer".

To Test H_0: $\mu = 15$
Against H_1: $\mu > 15$

Left tailed z-test at 5% level of significance

Under H_0, since the sample size, 110, is large, by Central Limit Theorem,

$\bar{T} \sim N\left(15, \dfrac{227.815}{110}\right)$ approximately

> **Commented [SH47]:** Recommendation
> Students are still unfamiliar with this topic. This is a show question so must working clearly for the calculation of the unbiased estimate of population variance.

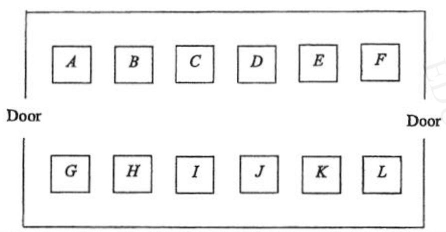
> **Commented [SH48]:** Question Reading
> In 3 dp not in 3 sf!

> **Commented [SH49]:** Presentation
> Majority of students wrongly stated the assumption as "by Central Limit Theorem, since the sample size is 110, which is more than 30 and is sufficiently large, $\bar{T}$ will be normally distributed". This is an approximation, not an assumption.
>
> So, students should pay attention and comment on the random variable T (which is the waiting time of a randomly chosen customer).

> **Commented [SH50]:** Presentation
> Careless use of Notations. Please pay attention to the use of the symbols as they would mean differently.

> **Commented [KSM51]:** Misconception
> Follow the logic of 'under Ho', the mean of $\bar{T}$ will be the pop mean value of 15!! Still many wrongly put it as the sample mean value.

> **Commented [SH52]:** Students use T instead of $\bar{T}$. The test is for the population mean not for the sample mean.

	From the G.C, p value = 0.0571 (Accept value of higher accuracy)	
	Since p-value = 0.0571 > 0.05, we do not reject H_0 and conclude that there is insufficient evidence that the mean waiting time is more than 15 minutes at 5% significance level.	
	(iii) Let X denote the waiting time of a randomly chosen customer.	
	Unbiased estimate of population variance $= \frac{n}{n-1}(2.5)^2 = \frac{6.25n}{n-1}$	
	To Test H_0: $\mu = 15$ Against H_1: $\mu \neq 15$	
	2-tailed z-test at 5% level of significance CLT	
	Under H_0, $\bar{X} \sim N\left(15, \frac{6.25}{n-1}\right)$ approx.	
	Since H_0 is rejected,	
	$\frac{15.6-15}{\left(\sqrt{\frac{6.25}{n-1}}\right)} < -1.96$ or $\frac{15.6-15}{\left(\sqrt{\frac{6.25}{n-1}}\right)} > 1.96$	
	$\sqrt{n-1} < -8.1665$ (no solution) or $\sqrt{n-1} > 8.1665$	
	$n > 67.692$	
	Least n is 68.	
9	A rectangular barrack with a door at each end, has 12 beds marked $A, B, C, \dots, L$ as shown in the diagram below. Johnson and 11 other soldiers take 1 bed each.	
		
	(i) Find the number of different sleeping arrangements possible if none of Johnson and 2 other particular soldiers are adjacent to each other, and all three of them are on the same side of the barrack.	[2]
	One afternoon, 9 particular soldiers went out to have lunch. There were 3 identical round tables at the dining venue.	
	(ii) How many ways can 9 soldiers be seated if there must be at least two soldiers at each table?	[4]

Commented [SH53]: Presentation
Students need to follow structure on the conclusion. Many of the students provided the conclusion on the basis of H_0. Which is incorrect and that is the reason also why many said there is sufficient evidence.

Commented [SH54]: Question Reading
Sample deviation is given implies sample variance is given. Almost all the students did not calculate the unbiased estimate for population variance.

Commented [SH55]: Presentation
Must show rejection at the left tail.

	Johnson was playing with a deck of 10 cards. When the cards are arranged in a certain manner, it forms the word SUCCESSFUL.	
	(iii) Find the number of ways he can arrange the 10 cards such that not all the letter "S" are together.	[2]
	(iv) He wishes to replace his existing handphone 4-letter codeword using letters from the word SUCCESSFUL, how many different options would he have.	[4]
	Solution	
	(i) Case 1:	
	The 3 individuals are in the top row. Since they must be separated, we will slot them among the 3 remaining people in that row. There are in total 4 possible slots. No. of ways $= \underbrace{{}^4C_3}_{\text{choose the 3 slots}} \times \underbrace{3!}_{\text{arrange among the 3 individuals}} \times \underbrace{9!}_{\text{arrange the other 9 people}}$ Case 2: The 3 individuals are in the bottom row. Same number of ways as Case 1.	
	Total number of ways $= 2\times\left[{}^4C_3\times3!\times9!\right] = 17418240$.	
	(ii) Case 1: 2, 2, 5 people No. of ways $= \left[\frac{{}^9C_2\times{}^7C_2}{2!}\times{}^5C_5\right]\times1!\times1!\times4! = 9072$	
	Case 2: 2, 3, 4 people No. of ways $= {}^9C_2\times{}^7C_3\times1!\times2!\times3! = 15120$	
	Case 3: 3, 3, 3 people No. of ways $= \frac{{}^9C_3\times{}^6C_3\times{}^3C_3}{3!}\times(2!)^3 = 2240$	
	Total no. of ways = 9072 + 15120 + 2240 = 26432	
	(iii) Number of ways = Number of ways without restrictions − number of ways that the SSS group is together $= \frac{10!}{2!\,2!\,3!} - \frac{8!}{2!\,2!}$	
	= 141120	

Commented [LMH56]: Things to note
Instead of writing 9! (to arrange all the other people), some students wrote 3! x 6! instead. These students have probably forgotten to choose who are the 3 soldiers who will be on the same side as Johnson and friends. They would be correct if they considered that, as ${}^9C_3\times3!\times6! = 9!$

Commented [LMH57]: Interpretation
Some students forgot the need to arrange them around the tables. Also, some students forgot to divide by 2! or 3! to account for the identical group sizes. Note that the tables are identical.

Commented [LMH58]: Question Reading
This is similar to the P&C question in MYCT. Yet significant number of students still couldn't understand "not all the 3 S are together" and proceeded to slot them.

If question wants you to separate all the 'S', this might be phrased as "no two S are together". In this case, yes, go ahead and use slotting method.

[Turn Over

BP

(iv)

Case I: 3 identical letters

Select 1 letter from U, C, E, F, L

No of 4-letter code-words with the chosen letter and 3 "S"s

$= {}^3C_1 \times \frac{4!}{3!} = 20$

Case II: 1 pair of identical letters

Select 1 pair from UU, CC, SS + select 2 distinct letters from remaining letters

No of 4-letter code-words with the chosen pair of same letters and the chosen 2 letters

$= {}^3C_1 \times {}^5C_2 \times \frac{4!}{2!} = 360$

Case III: 2 pairs of same letters

Select 2 pairs from UU, CC, SS and select 2 more distinct letters from remaining

No. of such code-words $= {}^3C_2 \times \frac{4!}{2!2!} = 18$

Case IV: All distinct letters

Select 4 letters from S, U, C, E, F, L

No. of such code-words $= {}^6C_4 \times 4! = 360$

Total no. of 4-letter code-words $= 20 + 360 + 18 + 360 = 758$

10 A ferry company operates a 9am ferry, with a passenger capacity of 108, from Island X to Island Y daily. To maximise profits, 113 tickets are sold online each day because it was found that on average, $p\%$ of customers who have purchased a ticket do not turn up.

(i) It is known that there is a probability of 0.012 that at most 1 customer will not turn up for the ferry. Write down an equation in terms of p, and hence find p correct to 3 decimal places. [3]

It is now given that $p = 6$.

(ii) Find the probability that there are no empty seats if every customer who turns up get a seat on the 9am ferry. [3]

(iii) Find the probability that, in a randomly chosen week, every customer who turns up gets a seat on the 9am ferry on at least 5 days of the week. [2]

(iv) Find the maximum number of tickets that could be sold so that the probability of a customer not getting a seat on the ferry is no more than 0.01. [3]

(v) A period of forty days are randomly selected. Find the probability that the mean daily number of customers who purchased a ticket and managed to turn up during this period is at most 106. [2]

Solution

(i)

Let X be the random variable denoting the number of customers who did not turn up for the 9 am ferry out of 113 customers on a randomly chosen day.

$X \sim B\left(113, \frac{p}{100}\right)$

Commented [LMH59]: Misconception
Note that when you use ${}^nC_r$, the objects you are choosing from must be **distinct**. It is clear that several students are not aware of this fact.

Commented [NCY60]: Presentation
Random variable should always be defined clearly and distribution must always be stated.

Commented [NCY61]: Question reading
P% → probability $= \frac{p}{100}$

Commented [LT62]: Question Reading
A number of students did not take note of the question requirement.

Commented [LT63]: Question Reading
Many did not read this part of the statement carefully and so failed to see that it is a conditional probability question.

Commented [LT64]: Misconception
Some students defined the random variable that is out of 108 customers.

Commented [LT65]: Misconception
The parameters for Binomial Distribution are number of trials and **probability of success**. So it is incorrect to write it as $X \sim B(113, p)$

It is also incorrect to write $p = \frac{p}{100}$ in the working as it would mean $1 = \frac{1}{100}$ for non-zero p.

$P(X \le 1) = 0.012$

$P(X=0) + P(X=1) = 0.012$

$\left(1-\frac{p}{100}\right)^{113} + 113\left(\frac{p}{100}\right)\left(1-\frac{p}{100}\right)^{112} = 0.012$

Using GC, $p = 5.554$ (3 decimal place)

(ii) Let Y be the random variable that denotes the number of customers who purchased a ticket and turn up for the 9 am ferry out of 113 customers.

$Y \sim B(113, 0.94)$

$P(Y \ge 108 \mid Y \le 108)$

$= \frac{P(Y=108)}{P(Y \le 108)}$

$= \frac{0.136717}{0.814523}$

$= 0.168$

(iii) Let W be the random variable that denotes the number of days where every customer who turns up get a seat on the 9 am ferry out of 7 days.

$W \sim B(7, 0.814523)$

$P(W \ge 5) = 1 - P(W \le 4) = 0.876$

(iv) Let T denote the number of customers who purchased a ticket and turn up for the 9am ferry out of n passengers.

$T \sim B(n, 0.94)$

$P(T > 108) \le 0.01$

$P(T \le 108) \ge 0.99$

Using GC,

When $n = 109$, $P(T \le 108) = 0.99882 > 0.99$

When $n = 110$, $P(T \le 108) = 0.99112 > 0.99$

When $n = 111$, $P(T \le 108) = 0.96571 < 0.99$

maximum $n = 110$

(v) Let Y be the random variable that denotes the number of customers who purchased a ticket and turn up for the 9 am ferry out of 113 customers.

$Y \sim B(113, 0.94)$

Since $n = 40$ is large, by Central Limit Theorem,

$\bar{Y} \sim N(106.22, 0.15933)$ approximately.

$P(\bar{Y} \le 106) = 0.291$

Commented [LT66]: Misconception
It is incorrect to apply the formula inside MF 26 directly. It is important to note that the probability has to be of the form $P(X = k)$ before applying the formula.

Commented [LT67]: Question Reading
Many cannot see that "every customer who turns up gets a seat" means $Y \le 108$.

Commented [LT68]: Accuracy of calculation
It is important to leave all intermediate answers correct to 5 significant figures.

Commented [LT69]: Notation
Do not reuse the same letter for different definitions.

Recommendation
Do not use the following letters:
B – For binomial distribution
P – For probability
O – Can be confused with origin or zero
N – Normal Distribution
Z – For Standard Normal
E – For Expectations

It is important to define the random variable properly for the subsequent workings to make sense.

Commented [LT70]: Question Reading
Many wrote down the incorrect probability statement.
For example
1. $P(T > 108) < 0.01$
2. $1 - P(T < 108) \le 0.01$

Commented [LT71]: Misconception
It is incorrect to write
$Y \sim N(106.22, 0.15933)$.

It is also important to note that the distribution obtained is an approximate Normal Distribution under Central Limit Theorem.

End of Paper