A Level H2 Math

Vectors Test 9

Q1

Referred to the origin O, the points A, B and C have position vectors **a**, **b** and **c** respectively such that

$$\mathbf{a} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}, \quad \mathbf{b} = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{c} = 4\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

(i) Given that M is the mid-point of AC, use a vector product to find the exact area of triangle ABM. [4]

(ii) Find the position vector of the point N on the line AB such that $\overrightarrow{MN}$ is perpendicular to $\overrightarrow{AB}$. [4]

Q2

The line l_1 has equation $\dfrac{x}{-3} = \dfrac{y}{12} = \dfrac{z-1}{4}$ and the line l_2 has equation $\dfrac{x-1}{-3} = y - 4 = \dfrac{z-1}{4}$.

(i) Show that l_1 and l_2 are skew lines. [3]

(ii) Find a cartesian equation of the plane p which is parallel to l_1 and contains l_2. [3]

(iii) The point $A(0, a, 1)$ is equidistant from p and l_1. Calculate the possible values of a exactly. [6]

Q3

It is given that $\mathbf{u} + \mathbf{v} - \mathbf{w}$ is perpendicular to $\mathbf{u} - \mathbf{v} + \mathbf{w}$, where **u**, **v** and **w** are unit vectors.

(i) Show that the angle between **v** and **w** is 60°. **[4]**

Referred to the origin O, the points U, V and W have position vectors **u**, **v** and **w** respectively.

(ii) Find the exact area of triangle OVW. **[2]**

(iii) Given that **u** and $\mathbf{v} \times \mathbf{w}$ are parallel, find the exact volume of the solid $OUVW$. **[2]**

[The volume of a pyramid is $\dfrac{1}{3}bh$, where b is the base area and h is the height of the pyramid.]

Answers
Vectors Test 9

Q1

(i) $\overrightarrow{OA}=\begin{pmatrix}2\\3\\-1\end{pmatrix}, \overrightarrow{OB}=\begin{pmatrix}5\\-2\\3\end{pmatrix}, \overrightarrow{OC}=\begin{pmatrix}4\\1\\-2\end{pmatrix}$

Since M is the mid-point of AC,

$$\overrightarrow{OM}=\frac{\begin{pmatrix}2\\3\\-1\end{pmatrix}+\begin{pmatrix}4\\1\\-2\end{pmatrix}}{2}=\frac{1}{2}\begin{pmatrix}6\\4\\-3\end{pmatrix}$$

$$\overrightarrow{AB}=\begin{pmatrix}5\\-2\\3\end{pmatrix}-\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}3\\-5\\4\end{pmatrix}$$

$$\overrightarrow{AM}=\frac{1}{2}\begin{pmatrix}6\\4\\-3\end{pmatrix}-\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}1\\-1\\-1/2\end{pmatrix}$$

Area of ΔABM

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$$=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AM}\right|$$

$$=\frac{1}{2}\left|\begin{pmatrix}3\\-5\\4\end{pmatrix}\times\begin{pmatrix}1\\-1\\-1/2\end{pmatrix}\right|$$

$$=\frac{1}{4}\left|\begin{pmatrix}3\\-5\\4\end{pmatrix}\times\begin{pmatrix}2\\-2\\-1\end{pmatrix}\right|=\frac{1}{4}\left|\begin{pmatrix}13\\11\\4\end{pmatrix}\right|=\frac{\sqrt{306}}{4}=\frac{3\sqrt{34}}{4}$$

Or:

$$\overrightarrow{AB}=\begin{pmatrix}5\\-2\\3\end{pmatrix}-\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}3\\-5\\4\end{pmatrix}, \overrightarrow{AC}=\begin{pmatrix}4\\1\\-2\end{pmatrix}-\begin{pmatrix}2\\3\\-1\end{pmatrix}=\begin{pmatrix}2\\-2\\-1\end{pmatrix}$$

Area of ΔABC

$$=\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right|$$

$$=\frac{1}{2}\left|\begin{pmatrix}3\\-5\\4\end{pmatrix}\times\begin{pmatrix}2\\-2\\-1\end{pmatrix}\right|$$

$$=\frac{1}{2}\left|\begin{pmatrix}13\\11\\4\end{pmatrix}\right|=\frac{\sqrt{306}}{2}$$

Area of ΔABM = $\frac{1}{2}$ Area of ΔABC = $\frac{\sqrt{306}}{4}=\frac{3\sqrt{34}}{4}$

(ii) $l_{AB}: \quad \mathbf{r} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}, \quad \lambda \in \mathbb{R}$

Since point N is on the line AB,

$$\overrightarrow{ON} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} \quad \text{for some } \lambda$$

$$\overrightarrow{MN} = \begin{pmatrix} 3\lambda - 1 \\ 1 - 5\lambda \\ \frac{1}{2} + 4\lambda \end{pmatrix}$$

For $\overrightarrow{MN}$ to be perpendicular to $\overrightarrow{AB}$,

$$\overrightarrow{MN} . \overrightarrow{AB} = 0$$

$$\begin{pmatrix} 3\lambda - 1 \\ 1 - 5\lambda \\ \frac{1}{2} + 4\lambda \end{pmatrix} . \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = 0$$

$$\lambda = \frac{3}{25}$$

$$\overrightarrow{ON} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} + \frac{3}{25} \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix} = \frac{1}{25} \begin{pmatrix} 59 \\ 60 \\ -13 \end{pmatrix}$$

Q2

(i)

$$l_1 : \mathbf{r} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 12 \\ 4 \end{pmatrix}, \ \lambda \in \mathbb{R} \qquad l_2 : \mathbf{r} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}, \ \mu \in \mathbb{R}$$

Since $\begin{pmatrix} -3 \\ 12 \\ 4 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$ are not parallel, l_1 and l_2 are not parallel.

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If the two lines intersect, there will be a unique value of λ and μ for the system of equations

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} -3 \\ 12 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix}$$

$$-3\lambda + 3\mu = 1 \quad (1)$$
$$12\lambda - \mu = 4 \quad (2)$$
$$4\lambda - 4\mu = 0 \quad (3)$$

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Using GC, no solution of λ and μ exist. Hence, the lines do not intersect.

Hence, l_1 and l_2 are skew lines.

(ii)

A normal to $p = \begin{pmatrix} -3 \\ 12 \\ 4 \end{pmatrix} \times \begin{pmatrix} -3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 44 \\ 0 \\ 33 \end{pmatrix} = 11 \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix}$

Since (1, 4, 1) lies on l_2 which is on p, $\begin{pmatrix} 1 \\ 4 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 0 \\ 3 \end{pmatrix} = 7$

Hence a cartesian equation for p is $4x + 3z = 7$.

(iii) (0, 0, 1) is a point on l_1.

$$\begin{pmatrix} 0 \\ a \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ a \\ 0 \end{pmatrix}$$

Distance from A to $l_1 = \dfrac{\left|\begin{pmatrix}0\\a\\0\end{pmatrix}\times\begin{pmatrix}-3\\12\\4\end{pmatrix}\right|}{\left|\begin{pmatrix}-3\\12\\4\end{pmatrix}\right|} = \dfrac{1}{13}\left|\begin{pmatrix}4a\\0\\3a\end{pmatrix}\right| = \dfrac{1}{13}\left|a\begin{pmatrix}4\\0\\3\end{pmatrix}\right| = \dfrac{5}{13}|a|$

(1, 4, 1) is a point on p.

$$\begin{pmatrix}0\\a\\1\end{pmatrix}-\begin{pmatrix}1\\4\\1\end{pmatrix}=\begin{pmatrix}-1\\a-4\\0\end{pmatrix}$$

Distance from A to $p = \dfrac{\left|\begin{pmatrix}-1\\a-4\\0\end{pmatrix}.\begin{pmatrix}4\\0\\3\end{pmatrix}\right|}{\left|\begin{pmatrix}4\\0\\3\end{pmatrix}\right|} = \dfrac{1}{5}|-4| = \dfrac{4}{5}$

Since the point A is equidistant to p and l_1,

$$\frac{5}{13}|a| = \frac{4}{5}$$

$$a = \pm\frac{52}{25}$$

Q3

(i)

Since $\mathbf{u}+\mathbf{v}-\mathbf{w}$ is perpendicular to $\mathbf{u}-\mathbf{v}+\mathbf{w}$,

$(\mathbf{u}+\mathbf{v}-\mathbf{w})\cdot(\mathbf{u}-\mathbf{v}+\mathbf{w})=0$

$\mathbf{u}\cdot\mathbf{u}-\mathbf{u}\cdot\mathbf{v}+\mathbf{u}\cdot\mathbf{w}$

$+\mathbf{v}\cdot\mathbf{u}-\mathbf{v}\cdot\mathbf{v}+\mathbf{v}\cdot\mathbf{w}$

$-\mathbf{w}\cdot\mathbf{u}+\mathbf{w}\cdot\mathbf{v}-\mathbf{w}\cdot\mathbf{w}=0$

Since $\mathbf{u}\cdot\mathbf{u}=|\mathbf{u}|^2, \mathbf{v}\cdot\mathbf{v}=|\mathbf{v}|^2, \mathbf{w}\cdot\mathbf{w}=|\mathbf{w}|^2$, and $\mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}, \mathbf{u}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{u}, \mathbf{v}\cdot\mathbf{w}=\mathbf{w}\cdot\mathbf{v}$,

$|\mathbf{u}|^2-|\mathbf{v}|^2-|\mathbf{w}|^2+2\mathbf{v}\cdot\mathbf{w}=0$

Since $\mathbf{u},\mathbf{v},\mathbf{w}$ are unit vectors, $|\mathbf{u}|=1, |\mathbf{v}|=1, |\mathbf{w}|=1$,

$1-1-1+2\mathbf{v}\cdot\mathbf{w}=0$

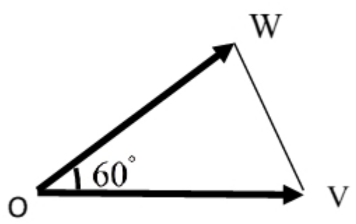
$$\mathbf{v}\cdot\mathbf{w}=\frac{1}{2}$$

$$|\mathbf{v}||\mathbf{w}|\cos\theta=\frac{1}{2}$$

$$\cos\theta=\frac{1}{2}$$

Hence, $\theta=60^\circ$

(ii)



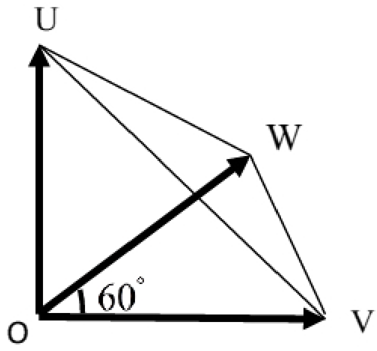
Area of $\triangle$OVW

$=\left(\frac{1}{2}(OV)(OW)\sin 60^\circ\right)$

$=\left(\frac{1}{2}\right)(1)(1)\left(\frac{\sqrt{3}}{2}\right)$

$=\frac{\sqrt{3}}{4}$ units^2

(iii)



Since $\mathbf{u}$ and $\mathbf{v}\times\mathbf{w}$ are parallel, we have $OU \perp OV, OU \perp OW$.

Volume of $OUVW$

$$=\frac{1}{3}(\text{Area of } \triangle \text{OVW})(OU)$$

$$=\frac{1}{3}\left(\frac{\sqrt{3}}{4}\right)(1)$$

$$=\frac{\sqrt{3}}{12} \text{ units}^3$$