A Level H2 Math

Vectors Test 8

Q1

The vectors **p** and **q** are given by $\mathbf{p} = 2\mathbf{i} + \mathbf{j} + a\mathbf{k}$ and $\mathbf{q} = b\mathbf{i} + \mathbf{j}$, where a and b are non-zero constants.

(i) Find $(2\mathbf{p} - 5\mathbf{q}) \cdot (2\mathbf{p} + 5\mathbf{q})$ in terms of a and b. [2]

Given that the **i**- and **j**- components of the answer to part **(i)** are equal, find the value of b. [1]

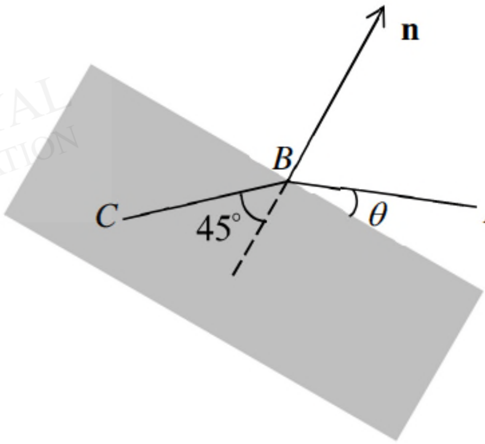
Use the value of b you have found to solve parts **(ii)** and **(iii)**.

(ii) Given that the magnitude of $(2\mathbf{p} - 5\mathbf{q}) \cdot (2\mathbf{p} + 5\mathbf{q})$ is 80, find the possible exact values of a. [2]

(iii) Given instead that $2\mathbf{p} - 5\mathbf{q}$ and $2\mathbf{p} + 5\mathbf{q}$ are perpendicular, find the exact value of $|\mathbf{p}|$. [3]

Q2

When a light ray passes from air to glass, it is deflected through an angle. The light ray ABC starts at point $A(1, 2, 2)$ and enters a glass object at point $B(0, 0, 2)$. The surface of the glass object is a plane with normal vector **n**. The diagram shows a cross-section of the glass object in the plane of the light ray and **n**.



(i) Find a vector equation of the line AB. [1]

The surface of the glass object is a plane with equation $x + z = 2$. AB makes an acute angle θ with the plane.

(ii) Calculate the value of θ, giving your answer in degrees. [2]

The line BC makes an angle of 45° with the normal to the plane, and BC is parallel to the unit vector $\begin{pmatrix} -\frac{2}{3} \\ p \\ q \end{pmatrix}$.

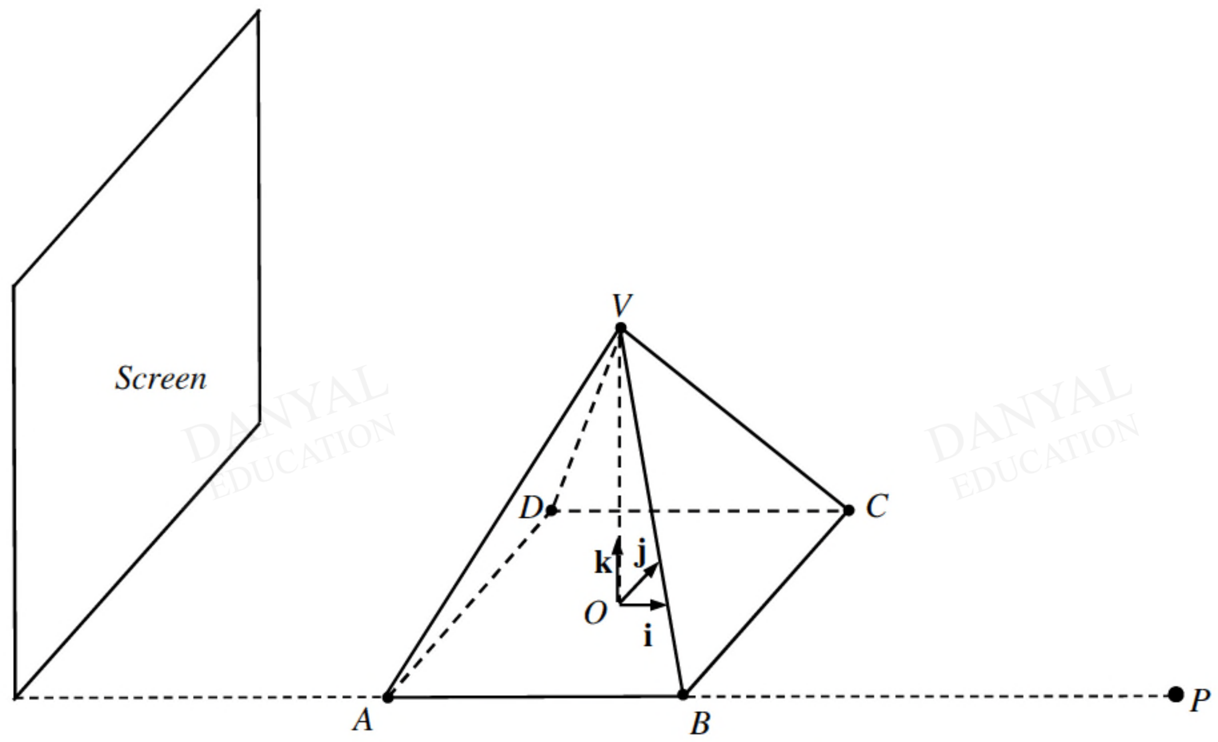
(iii) By considering a vector perpendicular to the plane containing the light ray and **n**, or otherwise, find the values of p and q. [4]

The light ray leaves the glass object through a plane with equation $3x + 3z = -4$.

(iv) Find the exact thickness of the glass object, taking one unit as one cm. [2]

(v) Find the exact coordinates of the point at which the light ray leaves the glass object. [3]

Q3



A right opaque pyramid with square base *ABCD* and vertex *V* is placed at ground level for a shadow display, as shown in the diagram. *O* is the centre of the square base *ABCD*, and perpendicular unit vectors **i**, **j**, **k** are in the directions of $\overrightarrow{AB}$, $\overrightarrow{AD}$ and $\overrightarrow{OV}$ respectively. The length of *AB* is 8 units and the length of *OV* is $2h$ units.

A point light source for this shadow display is placed at the point $P(20, -4, 0)$ and a screen of height 35 units is placed with its base on the ground such that the screen lies on a plane with vector equation $\mathbf{r}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}=\alpha$ where $\alpha < -4$ (see diagram).

(i) Find a vector equation of the line depicting the path of the light ray from *P* to *V* in terms of *h*. [2]

(ii) Find an inequality between α and *h* so that the shadow of the pyramid cast on the screen will not exceed the height of the screen. [3]

The point light source is now replaced by a parallel light source whose light rays are perpendicular to the screen and it is also given that $h = 10$.

(iii) Find the exact length of the shadow cast by the edge *VB* on the screen. [3]

A mirror is placed on the plane *VBC* to create a special effect during the display.

(iv) Find a vector equation of the plane *VBC* and hence find the angle of inclination made by the mirror with the ground. [4]

Answers
Vectors Test 8

Q1

(i)

$$
\begin{aligned}
(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q}) &= 4\mathbf{p}\times\mathbf{p}+10\mathbf{p}\times\mathbf{q}-10\mathbf{q}\times\mathbf{p}-25\mathbf{q}\times\mathbf{q} \\
&= 20\mathbf{p}\times\mathbf{q} \\
&= 20\begin{pmatrix}2\\1\\a\end{pmatrix}\times\begin{pmatrix}b\\1\\0\end{pmatrix} \\
&= 20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}
\end{aligned}
$$

Alternative:

$$
\begin{aligned}
(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q}) &= \left(2\begin{pmatrix}2\\1\\a\end{pmatrix}-5\begin{pmatrix}b\\1\\0\end{pmatrix}\right)\times\left(2\begin{pmatrix}2\\1\\a\end{pmatrix}+5\begin{pmatrix}b\\1\\0\end{pmatrix}\right) \\
&= \begin{pmatrix}4-5b\\-3\\2a\end{pmatrix}\times\begin{pmatrix}4+5b\\7\\2a\end{pmatrix} \\
&= \begin{pmatrix}-6a-14a\\-(8a-10ab-8a-10ab)\\28-35b+12+15b\end{pmatrix} \\
&= \begin{pmatrix}-20a\\20ab\\40-20b\end{pmatrix} = 20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}
\end{aligned}
$$

Given that the **i**- and **j**- components of the vector $20\begin{pmatrix}-a\\ab\\2-b\end{pmatrix}$ are equal,

$-a = ab$

$ab + a = 0$

$a(b+1) = 0$

Since $a \neq 0$, thus $b = -1$

(ii)

$$\left|(2\mathbf{p}-5\mathbf{q})\times(2\mathbf{p}+5\mathbf{q})\right|=80$$

$$\left|20\begin{pmatrix}-a\\ ab\\ 2-b\end{pmatrix}\right|=80$$

$$\left|\begin{pmatrix}-a\\ -a\\ 2+1\end{pmatrix}\right|=4$$

$$\sqrt{2a^2+9}=4$$

$$2a^2+9=16$$

$$a^2=\frac{7}{2}$$

$$a=\pm\sqrt{\frac{7}{2}}\ \ or\ \ \pm\frac{\sqrt{14}}{2}$$

(iii)

Since $2\mathbf{p}-5\mathbf{q}$ and $2\mathbf{p}+5\mathbf{q}$ are perpendicular,

$$(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q})=0$$

$$4|\mathbf{p}|^2-25|\mathbf{q}|^2=0$$

$$|\mathbf{p}|^2=\frac{25}{4}|\mathbf{q}|^2$$

$$=\frac{25}{4}\left((-1)^2+1^2\right)$$

$$=\frac{25}{2}$$

$$|\mathbf{p}|=\frac{5\sqrt{2}}{2}$$

Alternative:

$$(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q})=\begin{pmatrix}4+5\\ -3\\ 2a\end{pmatrix}\begin{pmatrix}4-5\\ 7\\ 2a\end{pmatrix}$$

$$=16-25-21+4a^2$$

$$=4a^2-30$$

Since $2\mathbf{p}-5\mathbf{q}$ and $2\mathbf{p}+5\mathbf{q}$ are perpendicular,

$$(2\mathbf{p}-5\mathbf{q})\cdot(2\mathbf{p}+5\mathbf{q})=0$$

$$4a^2-30=0$$

$$a^2=\frac{15}{2}$$

$$|\mathbf{p}|=\sqrt{2^2+1+a^2}=\sqrt{5+\frac{15}{2}}=\sqrt{\frac{25}{2}}=\frac{5\sqrt{2}}{2}$$

Q2

(i)

$$\overrightarrow{AB}=\begin{pmatrix}0\\0\\2\end{pmatrix}-\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}-1\\-2\\0\end{pmatrix}$$

$$l_{AB}:\underset{\sim}{r}=\begin{pmatrix}1\\2\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix}\quad or\quad \underset{\sim}{r}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\lambda\begin{pmatrix}1\\2\\0\end{pmatrix},\ \lambda\in\mathbb{R}\ \text{ or equivalent}$$

(ii)

$$\sin\theta=\frac{\left|\begin{pmatrix}1\\2\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\1\end{pmatrix}\right|}{\sqrt{5}\sqrt{2}}=\frac{1}{\sqrt{10}}$$

$$\theta=18.4^{\circ}$$

(iii)

Let $\underset{\sim}{m}$ be a vector perpendicular to the plane containing the light ray and $\underset{\sim}{n}$.

$$\underset{\sim}{m}=\underset{\sim}{n}\times\overrightarrow{AB}=\begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}-1\\-2\\0\end{pmatrix}=\begin{pmatrix}2\\-1\\-2\end{pmatrix}\quad \text{or}\quad \begin{pmatrix}1\\0\\1\end{pmatrix}\times\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}-2\\1\\2\end{pmatrix}$$

$$\cos 45^{\circ}=\frac{1}{\sqrt{2}}=\frac{\begin{pmatrix}-\frac{2}{3}\\p\\q\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\-1\end{pmatrix}}{\sqrt{2}}\ \Rightarrow\ \frac{2}{3}-q=1$$

$$q=-\frac{1}{3}$$

$$\begin{pmatrix}-\frac{2}{3}\\p\\q\end{pmatrix}\perp\underset{\sim}{m}\ \Rightarrow\ \begin{pmatrix}-\frac{2}{3}\\p\\-\frac{1}{3}\end{pmatrix}\cdot\begin{pmatrix}2\\-1\\-2\end{pmatrix}=0$$

$$-\frac{4}{3}-p+\frac{2}{3}=0\ \Rightarrow\ p=-\frac{2}{3}$$

(iv)
Glass upper surface is $x+z=2$

Glass bottom surface is $3x+3z=-4 \;\Rightarrow\; x+z=-\frac{4}{3}$

Distance between two planes $=\dfrac{\left|2-\left(-\frac{4}{3}\right)\right|}{\sqrt{2}}=\dfrac{10}{3\sqrt{2}}=\dfrac{5\sqrt{2}}{3}$

Thickness of the glass object is $\dfrac{5\sqrt{2}}{3}$ cm

(v)
Let the point at which the light ray leaves the glass object be F.

Method 1:

$$l_{BF}: \underset{\sim}{r}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\mu\begin{pmatrix}2\\2\\1\end{pmatrix} \quad or \quad \underset{\sim}{r}=\begin{pmatrix}0\\0\\2\end{pmatrix}+\mu\begin{pmatrix}-\frac{2}{3}\\-\frac{2}{3}\\-\frac{1}{3}\end{pmatrix}$$

At F,

$$\left[\begin{pmatrix}0\\0\\2\end{pmatrix}+\mu\begin{pmatrix}2\\2\\1\end{pmatrix}\right]\cdot\begin{pmatrix}3\\0\\3\end{pmatrix}=-4$$

$$6+\mu(6+3)=-4$$

$$\mu=-\frac{10}{9}$$

OR

$$\left[\begin{pmatrix}0\\0\\2\end{pmatrix}+\mu\begin{pmatrix}-\frac{2}{3}\\-\frac{2}{3}\\-\frac{1}{3}\end{pmatrix}\right]\cdot\begin{pmatrix}3\\0\\3\end{pmatrix}=-4$$

$$6+\mu(-2-1)=-4$$

$$\mu=\frac{10}{3}$$

The coordinates of F are

$$\left(-\frac{20}{9},-\frac{20}{9},\frac{8}{9}\right)$$

Method 2:

$$\cos 45^\circ=\frac{\frac{5\sqrt{2}}{3}}{BF} \;\Rightarrow\; \left|\vec{BF}\right|=\frac{5\sqrt{2}}{3}\times\sqrt{2}=\frac{10}{3}$$

(or using Pythagoras' theorem)

$$\vec{BF}=\frac{10}{3}\begin{pmatrix}-\frac{2}{3}\\-\frac{2}{3}\\-\frac{1}{3}\end{pmatrix}=-\frac{10}{9}\begin{pmatrix}2\\2\\1\end{pmatrix}$$

$$\vec{OF}=-\frac{10}{9}\begin{pmatrix}2\\2\\1\end{pmatrix}+\begin{pmatrix}0\\0\\2\end{pmatrix}=\frac{1}{9}\begin{pmatrix}-20\\-20\\8\end{pmatrix}$$

The coordinates of F are $\left(-\frac{20}{9},-\frac{20}{9},\frac{8}{9}\right)$

Q3

(i)

$$\overrightarrow{OP}=\begin{pmatrix}20\\-4\\0\end{pmatrix},\ \overrightarrow{OV}=\begin{pmatrix}0\\0\\2h\end{pmatrix}$$

$$\overrightarrow{PV}=\begin{pmatrix}-20\\4\\2h\end{pmatrix}=2\begin{pmatrix}-10\\2\\h\end{pmatrix}$$

Vector equation of the line depicting the path of the light ray from P to V is

$$\mathbf{r}=\begin{pmatrix}20\\-4\\0\end{pmatrix}+\lambda\begin{pmatrix}-10\\2\\h\end{pmatrix},\ \lambda\in\mathbb{R}$$

(ii) $\mathbf{r}.\begin{pmatrix}1\\0\\0\end{pmatrix}=\alpha$

$$\mathbf{r}=\begin{pmatrix}20\\-4\\0\end{pmatrix}+\lambda\begin{pmatrix}-10\\2\\h\end{pmatrix},\ \lambda\in\mathbb{R}$$

$$\begin{pmatrix}20-10\lambda\\-4+2\lambda\\\lambda h\end{pmatrix}.\begin{pmatrix}1\\0\\0\end{pmatrix}=\alpha$$

$$20-10\lambda=\alpha\Rightarrow\lambda=\frac{20-\alpha}{10}$$

For shadow of the pyramid cast on the screen to not exceed the height of the screen,

length of shadow, $\lambda h=\left(\frac{20-\alpha}{10}\right)h\le 35$

$\Rightarrow h\le\frac{350}{20-\alpha}$ since $\alpha<-4$ implies $20-\alpha>0$

(iii) Given that $h = 10$

$$\overrightarrow{OB} = \begin{pmatrix} 4 \\ -4 \\ 0 \end{pmatrix},\ \overrightarrow{OV} = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix} \Rightarrow \overrightarrow{BV} = \begin{pmatrix} -4 \\ 4 \\ 20 \end{pmatrix}$$

Length of the shadow cast by edge VB

$$= \left| \begin{pmatrix} -4 \\ 4 \\ 20 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right| = = \left| \begin{pmatrix} 0 \\ 20 \\ -4 \end{pmatrix} \right| = \sqrt{416} = 4\sqrt{26}$$

(iv)

$$\overrightarrow{OC} = \begin{pmatrix} 4 \\ 4 \\ 0 \end{pmatrix},\ \overrightarrow{OV} = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix},$$

$$\Rightarrow \overrightarrow{CV} = \begin{pmatrix} -4 \\ -4 \\ 20 \end{pmatrix} = -4\begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix}$$

$$\overrightarrow{BV} = \begin{pmatrix} -4 \\ 4 \\ 20 \end{pmatrix} = 4\begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix}$$

A vector normal to plane VBC is $\begin{pmatrix} 1 \\ 1 \\ -5 \end{pmatrix} \times \begin{pmatrix} -1 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 10 \\ 0 \\ 2 \end{pmatrix} = 2\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$

Vector equation of the plane VBC is

$$\mathbf{r}.\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 20 \end{pmatrix}.\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \mathbf{r}.\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} = 20$$

Angle of inclination made by the mirror with the ground

is $\cos^{-1}\left| \dfrac{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}}{\sqrt{1}\sqrt{25+1}} \right| = \cos^{-1}\left| \dfrac{1}{\sqrt{26}} \right| = 78.7^\circ$ (correct to 1 d.p.)