A Level H2 Math

Vectors Test 7

Q1

Planes Π_1 and Π_2 are defined by

$$\Pi_1 : x - 2y + 2z = 7, \quad \Pi_2 : \mathbf{r} \bullet \begin{pmatrix} a \\ 3 \\ -1 \end{pmatrix} = 8.$$

where a is a constant.

(i) The point P has position vector $-2\mathbf{i} + \mathbf{j} + \mathbf{k}$. Find the position vector of F, the foot of the perpendicular from P to plane Π_1.
Hence, or otherwise, find the shortest distance from P to plane Π_1. [5]

(ii) Line m passes through the point F and is parallel to both planes Π_1 and Π_2. Find the vector equation of line m. [2]

(iii) It is given that the point $Q(1,-4,-1)$ lies on line m. Find the value of a. [3]

(iv) Find the length of projection of $\overrightarrow{PQ}$ on the x-y plane. [3]

Q2

Referred to the origin O, the two points A and B have position vectors given by $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. The line l has equation $\mathbf{r} = 2\mathbf{a} + \lambda(\mathbf{a} + 2\mathbf{b})$, where $\lambda \in \mathbb{R}$. The point E is a general point on l and the point D has position vector $2\mathbf{a} - \mathbf{b}$.

Given that vector $\mathbf{a}$ is a unit vector, vector $\mathbf{b}$ has a magnitude of $\sqrt{2}$ units and that $\mathbf{a} \cdot \mathbf{b} = 1$,

(i) find the angle between vectors $\mathbf{a}$ and $\mathbf{b}$, and, [2]

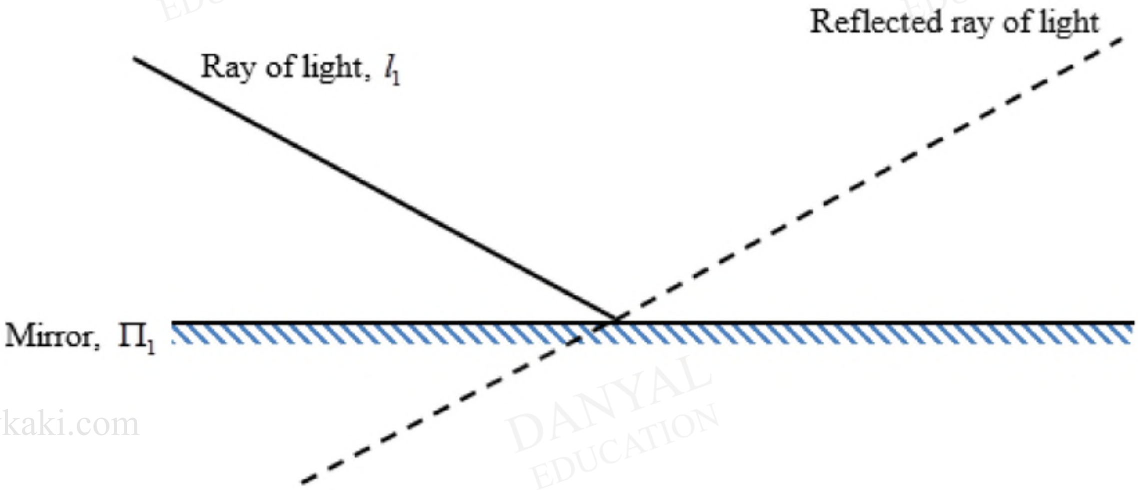
(ii) by considering $\overrightarrow{DE} \bullet \overrightarrow{DE}$, find an expression for the square of the distance DE, leaving your answer in terms of λ. [3]

Hence or otherwise, find the exact shortest distance of D to l, and write down the position vector of the foot of the perpendicular from D to l, in the form $p\mathbf{a} + q\mathbf{b}$. [3]

Q3

In the study of light, we may model a ray of light as a straight line.

A ray of light, l_1, is known to be parallel to the vector $2\mathbf{i}+\mathbf{k}$ and passes through the point P with coordinates $(1,1,0)$. The ray of light hits a mirror, and is reflected by the mirror which may be modelled by a plane Π_1 containing the points A, B and C with coordinates $(-1,1,0)$, $(0,0,2)$ and $(0,3,-3)$ respectively. This scenario is depicted in the diagram below:



(i) Show that an equation for plane Π_1 is given by $-x+5y+3z=6$. [3]

(ii) Find the coordinates of the point where the ray of light meets the mirror. [2]

(iii) Determine the position vector of the foot of the perpendicular from the point P to the mirror and hence, find an equation of the line that may be used to model the reflected ray of light. [6]

A second ray of light which is parallel to the mirror may be modelled by the line l_2, with Cartesian equation $\frac{x-1}{2}=\frac{z-2}{\alpha}, y=\beta$. Given that the distance between l_2 and the mirror is $\frac{14}{\sqrt{35}}$ units, find the values of the positive constants α and β. [4]

Answers
Vectors Test 7

Q1

(i) Equation of line through point P and perpendicular to π_1 is

$$\mathbf{r}=\begin{pmatrix}-2\\1\\1\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\2\end{pmatrix},\quad \lambda\in\mathbb{R}$$

Since F lies on plane π_1 ,

$(-2+\lambda) -2(1-2\lambda) + 2(1+2\lambda) = 7 \quad \Rightarrow \lambda = 1$

$$\overrightarrow{OF}=\begin{pmatrix}-2\\1\\1\end{pmatrix}+1\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}-1\\-1\\3\end{pmatrix}$$

$$\overrightarrow{PF}=\overrightarrow{OF}-\overrightarrow{OP}=\begin{pmatrix}-1\\-1\\3\end{pmatrix}-\begin{pmatrix}-2\\1\\1\end{pmatrix}=\begin{pmatrix}1\\-2\\2\end{pmatrix}$$

shortest distance from P to plane $\Pi_1=\left|\overrightarrow{PF}\right|=\sqrt{1^2+(-2)^2+2^2}=3$

(ii) Line m is parallel to both planes:

$$\begin{pmatrix}1\\-2\\2\end{pmatrix}\times\begin{pmatrix}a\\3\\-1\end{pmatrix}=\begin{pmatrix}2-6\\-(-1-2a)\\3+2a\end{pmatrix}=\begin{pmatrix}-4\\1+2a\\3+2a\end{pmatrix}$$

Equation of this line $m: r=\begin{pmatrix}-1\\-1\\3\end{pmatrix}+\mu\begin{pmatrix}-4\\1+2a\\3+2a\end{pmatrix}$ where $\mu\in\mathbb{R}$

(iii) $Q(1,-4,-1)$ lies on line m,

$-1-4\mu = 1$ --- (1)

$-1+(1+2a)\mu = -4$ --- (2)

$3+(3+2a)\mu = -1$ --- (3)

From (1) : $\mu = -\frac{1}{2}$
From (2) : $a = 5/2$
From (3) : $a = 5/2$.   Hence the value of a is 5/2

Alternative method

$$\overrightarrow{FQ} = \begin{pmatrix}1\\-4\\-1\end{pmatrix} - \begin{pmatrix}-1\\-1\\3\end{pmatrix} = \begin{pmatrix}2\\-3\\-4\end{pmatrix}$$

Since line m contains F and is parallel to π_1 , line m lies on π_1.

Since line m is on π_1, Q is on π_1. hence $\overrightarrow{FQ}$ is // π_1 and $\perp n_1$

$$\begin{pmatrix}2\\-3\\-4\end{pmatrix} \bullet \begin{pmatrix}a\\3\\-1\end{pmatrix} = 0$$

$$2a - 9 + 4 = 0$$

$$a = 5/2$$

(iv) Method 1 (dot product)

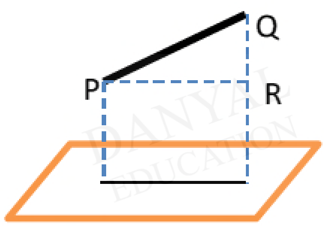
$$\overrightarrow{PQ} = \begin{pmatrix}1\\-4\\-1\end{pmatrix} - \begin{pmatrix}-2\\1\\1\end{pmatrix} = \begin{pmatrix}3\\-5\\-2\end{pmatrix} \text{ and normal to the } x\text{-}y \text{ plane} = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$QR = \left|\overrightarrow{PQ} \bullet \begin{pmatrix}0\\0\\1\end{pmatrix}\right| = \left|\begin{pmatrix}3\\-5\\-2\end{pmatrix} \bullet \begin{pmatrix}0\\0\\1\end{pmatrix}\right| = 2$$

length of projection of $\overrightarrow{PQ}$ on the x-y plane

= PR

$$= \sqrt{PQ^2 - 2^2} = \sqrt{(3^2 + 5^2 + 2^2 - 2^2} = \sqrt{34}$$



Method 2 (cross product)

length of projection of $\overrightarrow{PQ}$ on the x-y plane

$$= PR = \left|\overrightarrow{PQ} \times \begin{pmatrix}0\\0\\1\end{pmatrix}\right| = \left|\begin{pmatrix}3\\-5\\-2\end{pmatrix} \times \begin{pmatrix}0\\0\\1\end{pmatrix}\right| = \left|\begin{pmatrix}-5\\-3\\0\end{pmatrix}\right| = \sqrt{5^2 + 3^2} = \sqrt{34}$$

Q2

(i) $\mathbf{a}.\mathbf{b} = |a||b|\cos\theta \Rightarrow |1||\sqrt{2}|\cos\theta$

$\mathbf{a}.\mathbf{b} = 1 \quad \therefore \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = 45°$ (by inspection)

(ii) $\overline{DE} = \overline{OE} - \overline{OD} = 2\mathbf{a} + \lambda(\mathbf{a}+2\mathbf{b}) - (2\mathbf{a}-\mathbf{b}) = \mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b}),\ \lambda \in R$

To find the square of the distance DE

$DE^2 = [\mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b})].[\mathbf{b} + \lambda(\mathbf{a}+2\mathbf{b})]$

$= \mathbf{b}.\mathbf{b} + \lambda^2(\mathbf{a}+2\mathbf{b}).(\mathbf{a}+2\mathbf{b}) + 2\lambda\mathbf{b}.(\mathbf{a}+2\mathbf{b})$

$= \mathbf{b}.\mathbf{b} + \lambda^2(\mathbf{a}.\mathbf{a} + 4\mathbf{a}.\mathbf{b} + 4\mathbf{b}.\mathbf{b}) + 2\lambda(\mathbf{b}.\mathbf{a} + 2\mathbf{b}.\mathbf{b})$

$= 2 + \lambda^2(1 + 4(1) + 4(2)) + 2\lambda(1 + 2(2))$ as $\mathbf{a}.\mathbf{a} = 1$, $\mathbf{b}.\mathbf{b} = 2$ and $\mathbf{a}.\mathbf{b} = 1$

$= 2 + 13\lambda^2 + 10\lambda$

$= 13\lambda^2 + 10\lambda + 2$

(iii) **<u>Method One:</u>**

$DE^2 = 13\left[\lambda^2 + \frac{10}{13}\lambda\right] + 2$

$= 13\left(\lambda + \frac{10}{26}\right)^2 + 2 - \frac{25}{13} = 13\left(\lambda + \frac{5}{13}\right)^2 + \frac{1}{13}$

$DE = \sqrt{13\left(\lambda + \frac{5}{13}\right)^2 + \frac{1}{13}}$

The perpendicular distance from E to l occurs when D is closest to l, that is when DE is minimum or $\lambda = -\frac{5}{13}$.

Exact shortest distance from D to l is $\frac{1}{\sqrt{13}}$ units.

Method Two:

DE is minimum when DE^2 is minimum:

$$\frac{\mathrm{d}}{\mathrm{d}x}(DE^2) = 26\lambda + 10$$

To find stationary point:

When $\frac{\mathrm{d}}{\mathrm{d}x}(DE^2) = 0$, $26\lambda + 10 = 0$

$\therefore \lambda = -\frac{5}{13}$.

Since DE^2 is quadratic and coefficient of $\lambda^2 > 0$,

DE^2 is minimum at $\lambda = -\frac{5}{13}$

$\therefore$ perpendicular distance from D to l occur when $\lambda = -\frac{5}{13}$.

$$DE^2 = 13\lambda^2 + 10\lambda + 2 = 13\left(-\frac{5}{13}\right)^2 + 10\left(-\frac{5}{13}\right) + 2 = \frac{1}{13}$$

Exact shortest distance from D to l is $\frac{1}{\sqrt{13}}$ units.

(iv)

Let F be the foot of the perpendicular from D to l.

$$\overrightarrow{OF} = 2\mathbf{a} - \frac{5}{13}(\mathbf{a} + 2\mathbf{b}) = \frac{21}{13}\mathbf{a} - \frac{10}{13}\mathbf{b}$$

Q3

(i) $l_1 : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$

$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$; $\overrightarrow{AC} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$; $\overrightarrow{BC} = \begin{pmatrix} 0 \\ 3 \\ -3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \\ -5 \end{pmatrix}$

A normal to the plane is: $\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} \times \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}$

$\mathbf{r} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = 6$

Thus an equation for Π_1 is $-x + 5y + 3z = 6$. (shown)

(ii) Let N be the point of intersection between the line and the plane.

$\overrightarrow{ON} = \begin{pmatrix} 1+2\lambda \\ 1 \\ \lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$

Since N lies on the plane,

$\begin{pmatrix} 1+2\lambda \\ 1 \\ \lambda \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = 6 \Rightarrow \lambda = 2$

Thus, coordinates of N are (5, 1, 2).

(iii) Let the foot of the perpendicular from P to the plane be denoted by F.

$l_{PF} : \mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}, \mu \in \mathbb{R}$

Since F lies on l_{PF}, $\overrightarrow{OF} = \begin{pmatrix} 1-\mu \\ 1+5\mu \\ 3\mu \end{pmatrix}$ for some $\mu \in \mathbb{R}$

Since F lies on the plane, $\begin{pmatrix} 1-\mu \\ 1+5\mu \\ 3\mu \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = 6$

Solving, $\mu = \dfrac{2}{35}$

$$\overrightarrow{OF} = \begin{pmatrix} 33/35 \\ 9/7 \\ 6/35 \end{pmatrix}$$

Let the reflection of point P in the mirror be P'.

By the midpoint theorem, $\overrightarrow{OP'} = 2\overrightarrow{OF} - \overrightarrow{OP} = \begin{pmatrix} 31/35 \\ 11/7 \\ 12/35 \end{pmatrix}$

A direction vector for the reflected line is $\begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} - \begin{pmatrix} 31/35 \\ 11/7 \\ 12/35 \end{pmatrix} = \begin{pmatrix} 144/35 \\ -4/7 \\ 58/35 \end{pmatrix} = \frac{2}{35}\begin{pmatrix} 72 \\ -10 \\ 29 \end{pmatrix}$

Thus, an equation of the reflected line is:

$$l_1' : \mathbf{r} = \begin{pmatrix} 5 \\ 1 \\ 2 \end{pmatrix} + \gamma \begin{pmatrix} 72 \\ -10 \\ 29 \end{pmatrix}, \gamma \in \mathbb{R}$$

Since l_2 is parallel to Π_1, $\begin{pmatrix} 2 \\ 0 \\ \alpha \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix} = 0 \Rightarrow \alpha = \frac{2}{3}$

$$\begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 1 \\ \beta \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ -\beta \\ 0 \end{pmatrix}$$

Since the distance is $\frac{14}{\sqrt{35}}$, $\left| \frac{\begin{pmatrix} -1 \\ -\beta \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 5 \\ 3 \end{pmatrix}}{\sqrt{35}} \right| = \frac{14}{\sqrt{35}}$

$$|1 - 5\beta| = 14$$

Solving, $\beta = -\frac{13}{5}$ (rejected) or $\beta = 3$