A Level H2 Math

Vectors Test 6

Q1

With respect to the origin O, the position vectors of the points U, V and W are $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ respectively. The mid-points of the sides VW, WU and UV of the triangle UVW are M, N and P respectively.

(i) Show that $\overrightarrow{UM} = \frac{1}{2}(\mathbf{v} + \mathbf{w} - 2\mathbf{u})$. [2]

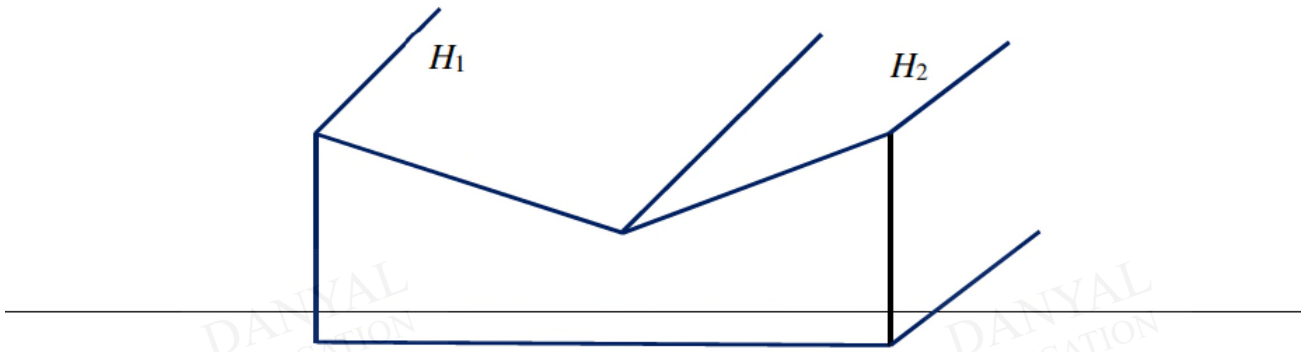
(ii) Find the vector equations of the lines UM and VN. Hence show that the position vector of the point of intersection, G, of UM and VN is $\frac{1}{3}(\mathbf{u} + \mathbf{v} + \mathbf{w})$. [5]

(iii) It is now given that $\mathbf{u} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\mathbf{v} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{w} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$. Find the direction cosines of $\overrightarrow{OG}$.

[2]

Q2

Peter is using equations of planes to model two hillsides that meet along a river. The river is modelled by the line where the two planes meet.



One of the hillsides, H_1, contains the points A, B and C with coordinates (3, 0, 2), (1, 0, 3) and $(2, -3, 5)$ respectively. The point A is on the river. The other hillside H_2 has equation $2x - y + kz = 14$, where k is a constant.

(i) Find a vector equation of H_1 in scalar product form. [4]

(ii) Show that $k = 4$ and deduce that point B is also on the river. [3]

(iii) Write down a cartesian equation of the river. [1]

(iv) Show that B is the point on the river that is nearest to C. Hence find the exact distance from C to the river. [3]

(v) Find the acute angle between BC and H_2. [2]

Q3

The position vectors of A, B and C with respect to the origin O are **a, b** and **c** respectively. It is given that $\overrightarrow{AC} = 4\overrightarrow{CB}$ and $|\mathbf{a}+\mathbf{b}|^2 = |\mathbf{a}|^2 + |\mathbf{b}|^2$.

(i) By considering $(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})$, show that **a** and **b** are perpendicular. [2]

(ii) Find the length of the projection of **c** on **a** in terms of $|\mathbf{a}|$. [3]

(iii) Given that F is the foot of the perpendicular from C to OA and **f** denotes the position vector $\overrightarrow{OF}$, state the geometrical meaning of $|\mathbf{c}\times\mathbf{f}|$. [1]

(iv) Two points X and Y move along line segments OA and AB respectively such that

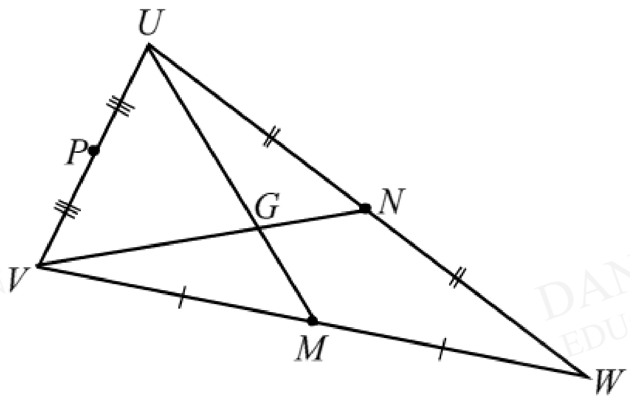
$$\overrightarrow{OX} = (\cos 3t)\mathbf{i} + (\sin 3t)\mathbf{j} + \frac{1}{2}\mathbf{k},$$

$$\overrightarrow{OY} = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} - 2\mathbf{k},$$

where t is a real parameter, $0 \le t \le 2\pi$. By expressing the scalar product of $\overrightarrow{OX}$ and $\overrightarrow{OY}$ in the form of $p\sin(qt)+r$ where p, q and r are real values to be determined, find the greatest value of the angle XOY. [5]

Answers
Vectors Test 6

Q1



(i)

By Ratio Theorem, $\overrightarrow{UM} = \dfrac{\overrightarrow{UW}+\overrightarrow{UV}}{2}$

$= \dfrac{\mathbf{w}-\mathbf{u}+\mathbf{v}-\mathbf{u}}{2}$

$= \underline{\underline{\dfrac{1}{2}(\mathbf{v}+\mathbf{w}-2\mathbf{u})}}$ (Shown)

(ii) Vector equation of line UM is $\underline{\underline{\mathbf{r}=\mathbf{u}+\lambda(\mathbf{w}+\mathbf{v}-2\mathbf{u})}}$, $\lambda \in \mathbb{R}$

$\overrightarrow{VN} = \dfrac{\overrightarrow{VW}+\overrightarrow{VU}}{2}$

$= \dfrac{\mathbf{w}-\mathbf{v}+\mathbf{u}-\mathbf{v}}{2} = \dfrac{1}{2}(\mathbf{w}+\mathbf{u}-2\mathbf{v})$

Vector equation of line VN is $\underline{\underline{\mathbf{r}=\mathbf{v}+\mu(\mathbf{w}+\mathbf{u}-2\mathbf{v})}}$, $\mu \in \mathbb{R}$

At point of intersection G,

$\mathbf{u}+\lambda(\mathbf{w}+\mathbf{v}-2\mathbf{u}) = \mathbf{v}+\mu(\mathbf{w}+\mathbf{u}-2\mathbf{v})$

For $\mathbf{u}$: $1-2\lambda = \mu$

For $\mathbf{w}$: $\lambda = \mu$

Solving, $\lambda = \dfrac{1}{3} = \mu$

$$\overrightarrow{OG} = \mathbf{u}+\frac{1}{3}(\mathbf{w}+\mathbf{v}-2\mathbf{u})$$
$$= \frac{1}{3}(\mathbf{u}+\mathbf{v}+\mathbf{w}) \quad \text{(Shown)}$$

(iii) $\mathbf{u}=\begin{pmatrix}1\\0\\0\end{pmatrix}, \mathbf{v}=\begin{pmatrix}0\\1\\0\end{pmatrix}, \mathbf{w}=\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$\overrightarrow{OG} = \frac{1}{3}\left[\begin{pmatrix}1\\0\\0\end{pmatrix}+\begin{pmatrix}0\\1\\0\end{pmatrix}+\begin{pmatrix}0\\0\\1\end{pmatrix}\right]=\begin{pmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{pmatrix}$$

$$\left|\overrightarrow{OG}\right| = \sqrt{3\left(\frac{1}{3^2}\right)}=\sqrt{\frac{1}{3}}$$

Direction cosines of $\overrightarrow{OG}$ are $\frac{\frac{1}{3}}{\sqrt{\frac{1}{3}}}, \frac{\frac{1}{3}}{\sqrt{\frac{1}{3}}}, \frac{\frac{1}{3}}{\sqrt{\frac{1}{3}}}$, i.e., $\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}$

Q2

(i) $A(3, 0, 2), B\ (1, 0, 3), C(2, -3, 5)$

$$\overrightarrow{AB} = \begin{pmatrix}1\\0\\3\end{pmatrix} - \begin{pmatrix}3\\0\\2\end{pmatrix} = \begin{pmatrix}-2\\0\\1\end{pmatrix} \qquad \overrightarrow{AC} = \begin{pmatrix}2\\-3\\5\end{pmatrix} - \begin{pmatrix}3\\0\\2\end{pmatrix} = \begin{pmatrix}-1\\-3\\3\end{pmatrix}$$

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix}-2\\0\\1\end{pmatrix} \times \begin{pmatrix}-1\\-3\\3\end{pmatrix} = \begin{pmatrix}3\\5\\6\end{pmatrix}$$

Take $\mathbf{n}_1 = \begin{pmatrix}3\\5\\6\end{pmatrix}$, $\mathbf{a} \bullet \mathbf{n}_1 = \begin{pmatrix}1\\0\\3\end{pmatrix} \bullet \begin{pmatrix}3\\5\\6\end{pmatrix} = 3+0+18 = 21$

A vector equation of H_1 is $\mathbf{r} \bullet \begin{pmatrix}3\\5\\6\end{pmatrix} = 21$

(ii) Equation of H_2 is $2x - y + kz = 14$.

Sub. $A(3, 0, 2)$ into equation of H_2,

$2(3) - 0 + k(2) = 14$

$\therefore\ k = 4$ (Shown)

Sub. $B(1, 0, 3)$ into LHS of equation of H_2,

LHS $= 2x - y + 4z = 2(1) - 0 + 4(3) = 14 =$ RHS

$\therefore\ B$ is also in H_2.

Since B is in both H_1 and H_2, $\therefore\ B$ is on the river. (Deduced)

(iii) Recall $\overrightarrow{AB} = \begin{pmatrix}-2\\0\\1\end{pmatrix}$, using $A(3, 0, 2)$ or $B\ (1, 0, 3)$,

a cartesian equation of the river (line AB) is

$$\frac{x-3}{-2} = z-2,\ y=0 \quad \text{or} \quad \frac{x-1}{-2} = z-3,\ y=0$$

(iv) Since $\overrightarrow{BC} \bullet \overrightarrow{AB} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} = 1(-2) + (-3)(0) + 2(1) = 0$,

BC is perpendicular to AB.

$\therefore$ B is the point on the river that is nearest to C.

Exact distance from C to the river

$$= \left|\overrightarrow{BC}\right| = \left|\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix}\right| = \sqrt{1+9+4} = \underline{\underline{\sqrt{14}}}$$

(v) Acute angle between BC and H_2

$$\theta = \sin^{-1} \frac{\left|\begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} \bullet \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix}\right|}{\sqrt{14}\sqrt{21}} = \sin^{-1} \frac{13}{\sqrt{14}\sqrt{21}}$$

$$= \underline{\underline{49.3^\circ}} \text{ or } \underline{\underline{0.861 \text{ rad}}}$$

Q3

9(i)

$(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=\mathbf{a.a}+2\mathbf{a.b}+\mathbf{b.b}$

Since $(\mathbf{a}+\mathbf{b}).(\mathbf{a}+\mathbf{b})=|\mathbf{a}+\mathbf{b}|^2$

and given $|\mathbf{a}+\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2$

$\therefore\ \mathbf{a.a}+2\mathbf{a.b}+\mathbf{b.b}=|\mathbf{a}|^2+|\mathbf{b}|^2$

$|\mathbf{a}|^2+2\mathbf{a.b}+|\mathbf{b}|^2=|\mathbf{a}|^2+|\mathbf{b}|^2$

$2\mathbf{a.b}=0$

$\mathbf{a.b}=0$

$\therefore\ \mathbf{a}\perp\mathbf{b}$

ii)

Using ratio theorem, $\overrightarrow{OC}=\frac{4\mathbf{b}+\mathbf{a}}{5}=\frac{1}{5}\mathbf{a}+\frac{4}{5}\mathbf{b}$.

Length of projection of $\overrightarrow{\mathbf{OC}}$ onto $\overrightarrow{\mathbf{OA}}$

$$=\frac{\left|\overrightarrow{OC}\bullet\overrightarrow{OA}\right|}{\left|\overrightarrow{OA}\right|}$$

$$=\frac{\left|\left(\frac{1}{5}\mathbf{a}+\frac{4}{5}\mathbf{b}\right)\bullet\mathbf{a}\right|}{|\mathbf{a}|}=\frac{\left|\frac{1}{5}\mathbf{a}\bullet\mathbf{a}+\frac{4}{5}\mathbf{b}\bullet\mathbf{a}\right|}{|\mathbf{a}|}$$

$$=\frac{\left|\frac{1}{5}|\mathbf{a}|^2+\frac{4}{5}\mathbf{b}\bullet\mathbf{a}\right|}{|\mathbf{a}|}=\frac{1}{5}|\mathbf{a}| \qquad \textbf{(since } \mathbf{a}\perp\mathbf{b}\textbf{)}$$

iii)

$|\mathbf{c}\times\mathbf{f}|$ denotes twice the area of the triangle COF.

(iv)

$$\overrightarrow{OX}\bullet\overrightarrow{OY}=\begin{pmatrix}\cos 3t\\ \sin 3t\\ \frac{1}{2}\end{pmatrix}\bullet\begin{pmatrix}\sin t\\ \cos t\\ -2\end{pmatrix}=\cos 3t\sin t+\sin 3t\cos t-1=\sin(4t)-1$$

$$\cos\measuredangle XOY=\frac{\overrightarrow{OX}\bullet\overrightarrow{OY}}{\left|\overrightarrow{OX}\right|\left|\overrightarrow{OY}\right|}=\frac{\sin 4t-1}{\sqrt{\cos^2 3t+\sin^2 3t+\frac{1}{4}}\sqrt{\sin^2 t+\cos^2 t+4}}$$

$$=\frac{\sin 4t-1}{\sqrt{\frac{5}{4}}\sqrt{5}}$$

$$=\frac{2}{5}(\sin 4t-1)$$

Maximum $\measuredangle XOY$ occurs when is most negative.
i.e. when $\sin 4t=-1$.
At that value of t,

$$\cos\measuredangle XOY=\frac{2}{5}(-1-1)=-\frac{4}{5}$$

$$\therefore\measuredangle XOY=\cos^{-1}\left(-\frac{4}{5}\right)=143.1^\circ$$