**A Level H2 Math**

**Vectors Test 5**

Q1

Referred to the origin $O$, the points $A$ and $B$ have position vectors $\mathbf{a}$ and $\mathbf{b}$, where $\mathbf{a}$ and $\mathbf{b}$ are non-zero and non-parallel. The point $C$ lies on $OB$ produced such that $3OC = 5OB$. It is given that $|\mathbf{a}| = 2|\mathbf{b}|$ and $\cos\angle AOB = -\frac{1}{4}$.

**(a)** **(i)** Show that a vector equation of the line $AC$ is $\mathbf{r} = \mathbf{a} + \lambda(3\mathbf{a} - 5\mathbf{b})$, where $\lambda$ is a real parameter. [2]

The line $l$ lies in the plane containing $O$, $A$ and $B$.

**(ii)** Explain why the direction vector of $l$ can be expressed as $s\mathbf{a} + t\mathbf{b}$, where $s$ and $t$ are real numbers. [1]

Given that $l$ is perpendicular to $AB$, show that $t = 3s$. [4]

Given further that $l$ passes through $B$, write down a vector equation of $l$, in a similar form as part **(i)**. [1]

**(iii)** Find the position vector of the point of intersection of line $AC$ and $l$, in terms of $\mathbf{a}$ and $\mathbf{b}$. [2]

**(b)** Explain why, for any constant $k$, $|(\mathbf{a} + k\,\mathbf{b}) \times \mathbf{b}|$ gives the area of the parallelogram with sides $OA$ and $OB$. Find the area of the parallelogram, leaving your answer in terms of $|\mathbf{a}|$. [4]

Q2

**(a)** It is given that three non-zero vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ satisfy the equation $(\mathbf{a}+\mathbf{b})\times(\mathbf{a}+\mathbf{c})=\mathbf{b}\times\mathbf{c}$, where $\mathbf{b}\neq\mathbf{c}$. Find a linear relationship between $\mathbf{a}$, $\mathbf{b}$ an $\mathbf{c}$. [3]

**(b)** A point $A$ with position vector $\overrightarrow{OA}=\alpha\,\mathbf{i}+\beta\,\mathbf{j}+\gamma\,\mathbf{k}$, where $\alpha$, $\beta$ and $\gamma$ are real constants, has direction cosines $\cos\theta$, $\cos\phi$ and $\cos\omega$, where $\theta$, $\phi$ and $\omega$ are the angles $\overrightarrow{OA}$ make with the positive $x$, $y$ and $z$ -axes respectively.

**(i)** Express the direction cosines $\cos\theta$, $\cos\phi$ and $\cos\omega$ in terms of $\alpha$, $\beta$ and $\gamma$.

Hence find the value of $\cos^2\theta+\cos^2\phi+\cos^2\omega$. [3]

**(ii)** Hence show that $\cos 2\theta+\cos 2\phi+\cos 2\omega=-1$. [2]

Q3

Referred to the origin $O$, the position vector of a point $A$ is $-\mathbf{i}+2\mathbf{j}+4\mathbf{k}$. A plane $p$ contains $A$ and is parallel to the vectors $4\mathbf{i}-2\mathbf{j}+\mathbf{k}$ and $2\mathbf{i}+\mathbf{k}$.

**(i)** Find a cartesian equation of $p$. [2]

**(ii)** A plane $q$ has equation $x-2y+z=2$. Find a vector equation of the line $l$ where $p$ and $q$ meet. [1]

A point $B$ lies on $l$ such that $AB$ is perpendicular to $l$.

**(iii)** Find the position vector of $B$. [3]

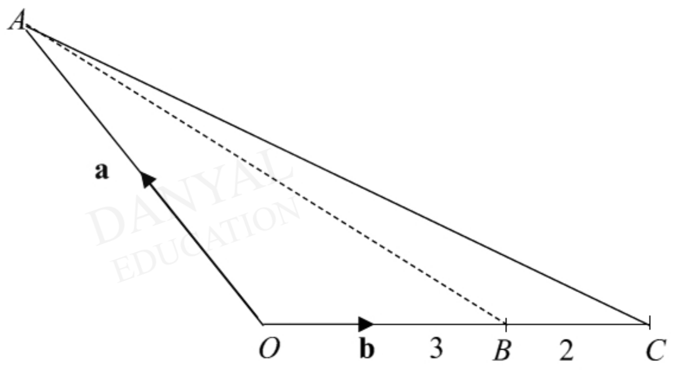
**(iv)** Find the length of projection of $AB$ on $q$. [2]

**(v)** A point $C$ lies on $q$ such that $AC$ is perpendicular to $q$. Find the position vector of $C$. Hence find a cartesian equation of the line of reflection of $AB$ in $q$. [6]

**Answers**
**Vectors Test 5**

Q1

(a)




(i) $\overrightarrow{OC} = \frac{5}{3}\mathbf{b} \quad \therefore \overrightarrow{AC} = \frac{5}{3}\mathbf{b} - \mathbf{a} = \frac{1}{3}(5\mathbf{b} - 3\mathbf{a}) \,//\, 5\mathbf{b} - 3\mathbf{a}$

Equation of line $AC$: $\mathbf{r} = \mathbf{a} + \lambda(3\mathbf{a} - 5\mathbf{b}), \quad \lambda \in \mathbb{R}$

(ii) Since $l$ lies on the plane containing $O$, $A$ and $B$, its direction vector is coplanar with a and b, thus it will be a linear combination of a and b, i.e. $s\mathbf{a} + t\mathbf{b}$ is a direction vector for $l$.

(iii) Let intersection point be $D$.

At $D$,

$\mathbf{a} + \lambda(3\mathbf{a} - 5\mathbf{b}) = \mathbf{b} + \mu(\mathbf{a} + 3\mathbf{b})$

Since a and b are non-zero, non-parallel vectors,

$1 + 3\lambda = \mu \text{------}(1)$

$-5\lambda = 1 + 3\mu \text{-----}(2)$

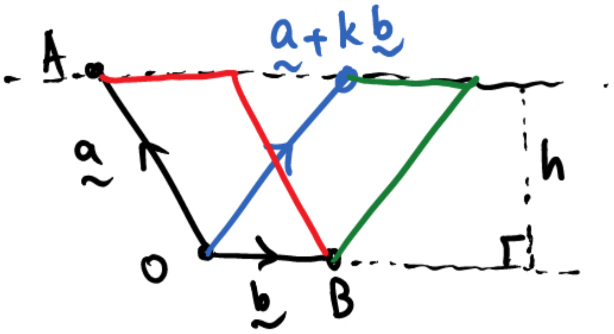
Solving,

$-5\lambda = 1 + 3(1 + 3\lambda)$

$14\lambda = -4$

$\therefore \lambda = -\frac{2}{7}, \mu = \frac{1}{7} \qquad \therefore \overrightarrow{OD} = \frac{1}{7}\mathbf{a} + \frac{10}{7}\mathbf{b}$

(b)



Method 1

Since the base length ($OB$) and perpendicular height remain the same, the area of parallelograms formed by different $k$ remains the same as the area of the parallelogram with sides $OA$ and $OB$.

Method 2

$$\left|(\mathbf{a}+k\mathbf{b})\times\mathbf{b}\right| = |\mathbf{a}\times\mathbf{b}+k\mathbf{b}\times\mathbf{b}| = |\mathbf{a}\times\mathbf{b}+\mathbf{0}| = |\mathbf{a}\times\mathbf{b}|$$

Area of parallelogram

$$= \left|\mathbf{a}\times\mathbf{b}\right| = |\mathbf{a}||\mathbf{b}|\sin\theta$$

$$= |\mathbf{a}|\left(\frac{1}{2}|\mathbf{a}|\right)\sqrt{1-\left(-\frac{1}{4}\right)^2}$$

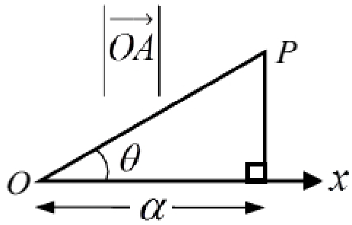
$$= \frac{\sqrt{15}}{8}|\mathbf{a}|^2$$

Q2
**(a)**

$$(\underset{\sim}{a}+\underset{\sim}{b})\times(\underset{\sim}{a}+\underset{\sim}{c})=\underset{\sim}{b}\times\underset{\sim}{c}$$
$$(\underset{\sim}{a}\times\underset{\sim}{a})+(\underset{\sim}{a}\times\underset{\sim}{c})+(\underset{\sim}{b}\times\underset{\sim}{a})+(\underset{\sim}{b}\times\underset{\sim}{c})=\underset{\sim}{b}\times\underset{\sim}{c}$$
$$(\underset{\sim}{a}\times\underset{\sim}{c})+(\underset{\sim}{b}\times\underset{\sim}{a})=\underset{\sim}{0}$$
$$(\underset{\sim}{a}\times\underset{\sim}{c})-(\underset{\sim}{a}\times\underset{\sim}{b})=\underset{\sim}{0}$$
$$\underset{\sim}{a}\times(\underset{\sim}{c}-\underset{\sim}{b})=\underset{\sim}{0}$$

Since $\underset{\sim}{a}$ is non-zero and $\underset{\sim}{b}\neq\underset{\sim}{c}$,
$\therefore$ $\underset{\sim}{a}$ is parallel to $(\underset{\sim}{c}-\underset{\sim}{b})$.
$\therefore$ $\underset{\sim}{a}=k(\underset{\sim}{c}-\underset{\sim}{b})$, $k\in\mathbb{R}$.

**(b)(i)**




$$\left|\overrightarrow{OA}\right|=\sqrt{\alpha^2+\beta^2+\gamma^2}$$

$$\therefore\quad \cos\theta=\frac{\alpha}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$$
$$\cos\phi=\frac{\beta}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$$
$$\cos\omega=\frac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}}$$

$$\cos^2\theta+\cos^2\phi+\cos^2\omega$$
$$=\left(\frac{\alpha}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\right)^2+\left(\frac{\beta}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\right)^2+\left(\frac{\gamma}{\sqrt{\alpha^2+\beta^2+\gamma^2}}\right)^2$$
$$=\frac{\alpha^2+\beta^2+\gamma^2}{\alpha^2+\beta^2+\gamma^2}$$
$$=1$$

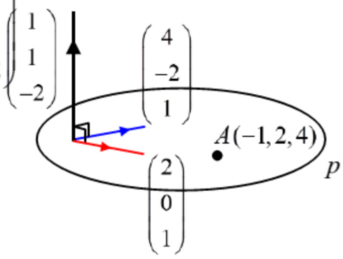
**(b)(ii)**

$$\cos2\theta+\cos2\phi+\cos2\omega$$
$$=2\cos^2\theta-1+2\cos^2\phi-1+2\cos^2\omega-1$$
$$=2\left(\cos^2\theta+\cos^2\phi+\cos^2\omega\right)-3$$
$$=2(1)-3$$
$$=-1 \quad \text{(shown)}$$

Q3

**(i)** $\begin{pmatrix}4\\-2\\1\end{pmatrix}\times\begin{pmatrix}2\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\-2\\4\end{pmatrix}=-2\begin{pmatrix}1\\1\\-2\end{pmatrix}$

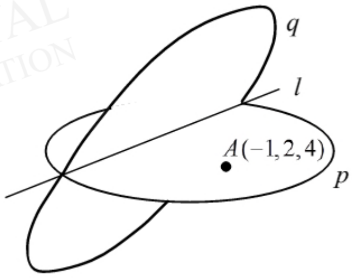
$\underset{\sim}{r}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=\begin{pmatrix}-1\\2\\4\end{pmatrix}\cdot\begin{pmatrix}1\\1\\-2\end{pmatrix}=-7$




$\therefore$ Cartesian equation of $p$ is $x+y-2z=-7$.

**(ii)** $x+y-2z=-7$
$x-2y+z=2$
Using GC,
a vector equation of $l$ is

$\underset{\sim}{r}=\begin{pmatrix}-4\\-3\\0\end{pmatrix}+\alpha\begin{pmatrix}1\\1\\1\end{pmatrix},\alpha\in\mathbb{R}.$

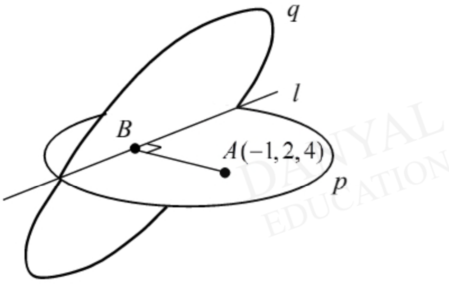



**(iii)**

$\overrightarrow{OB}=\begin{pmatrix}-4+\alpha\\-3+\alpha\\\alpha\end{pmatrix}$

$\overrightarrow{AB}=\begin{pmatrix}-4+\alpha\\-3+\alpha\\\alpha\end{pmatrix}-\begin{pmatrix}-1\\2\\4\end{pmatrix}$

$=\begin{pmatrix}\alpha-3\\\alpha-5\\\alpha-4\end{pmatrix}$



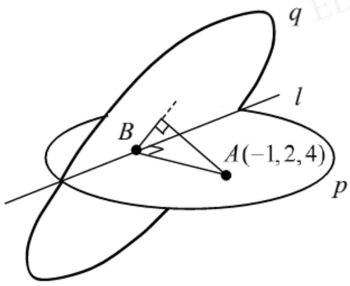

$\begin{pmatrix}\alpha-3\\\alpha-5\\\alpha-4\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=0$

$\alpha-3+\alpha-5+\alpha-4=0 \quad\Rightarrow\quad \alpha=4$

$\therefore\ \overrightarrow{OB}=\begin{pmatrix}0\\1\\4\end{pmatrix}=\underset{\sim}{j}+4\underset{\sim}{k}$

**(iv)** Equation of $q$: $\underset{\sim}{r}\cdot\begin{pmatrix}1\\-2\\1\end{pmatrix}=2$

$\overrightarrow{AB}=\begin{pmatrix}4-3\\4-5\\4-4\end{pmatrix}=\begin{pmatrix}1\\-1\\0\end{pmatrix}$




$\therefore$ length of projection of $AB$ on $q$ is

$\left|\overrightarrow{AB}\times\frac{1}{\sqrt{6}}\begin{pmatrix}1\\-2\\1\end{pmatrix}\right|=\frac{1}{\sqrt{6}}\left|\begin{pmatrix}1\\-1\\0\end{pmatrix}\times\begin{pmatrix}1\\-2\\1\end{pmatrix}\right|=\frac{1}{\sqrt{6}}\left|\begin{pmatrix}-1\\-1\\-1\end{pmatrix}\right|=\frac{1}{\sqrt{6}}\left(\sqrt{3}\right)=\frac{\sqrt{2}}{2}$