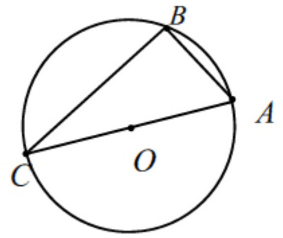
A Level H2 Math

Vectors Test 4

Q1



The diagram above shows the cross-section of a sphere containing the centre O of the sphere. The points A, B and C are on the circumference of the cross-section with the line segment AC passing through O. It is given that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

(i) Find $\overrightarrow{BC}$ in terms of **a** and **b**. [1]

(ii) D is a point on BC such that ΔOCD is similar to ΔACB. Find $\overrightarrow{OD}$ in terms of **a** and **b**. [2]

Point B lies on the x-z plane and has a positive z-component. It is also given that $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ and $\angle OCB = \frac{\pi}{6}$.

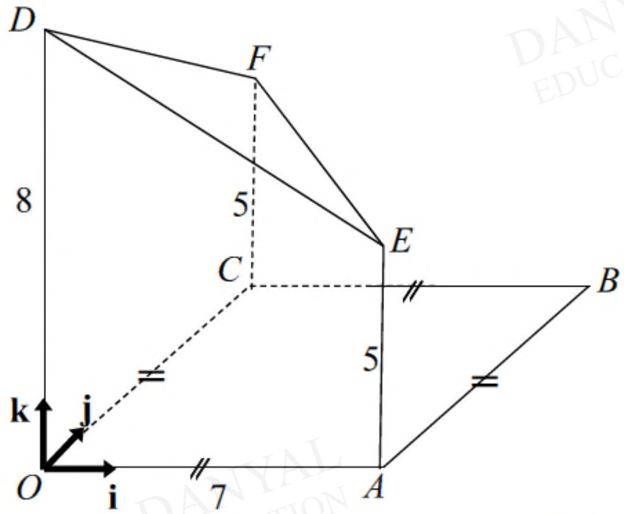
(iii) Show that $\overrightarrow{OB} = \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix}$. [4]

(iv) Hence, determine whether the line passing through O and B and the line $\frac{x-2}{3} = \frac{y}{3} = z - 1$ are skew. [3]

Q2

The diagram (not drawn to scale) shows the structure of a partially constructed building that is built on a horizontal ground. The building has a square base foundation of 7 m in length. Points *O*, *A*, *B* and *C* are the corners of the foundation of the building. The building currently consists of three vertical pillars *OD*, *AE* and *CF* of heights 8 m, 5 m and 5 m respectively.

A canvas is currently attached at *D*, *E* and *F*, forming a temporary shelter for the building. *O* is taken as the origin and vectors **i**, **j**, and **k**, each of length 1 m, are taken along *OA*, *OC* and *OD* respectively.



(i) Find a Cartesian equation of the plane that represents the canvas *DEF*. [3]

(ii) Find the acute angle which the canvas *DEF* makes with the horizontal ground. [2]

(iii) Given that the canvas is to be extended along the plane *DEF* till it touches the horizontal ground, explain why point *B* will lie beneath the canvas. [2]

A cement roof is to be built to replace the extended canvas. A vertical partition wall is also to be built such that it is *d* m away from and parallel to the plane *ODFC*, where $0 < d < 7$.

(iv) Find the exact vector equation of the line where the roof meets the partition wall. Show your working clearly, leaving your answer in terms of *d*. [4]

(v) A lighting point, *P*, is to be placed on the roof such that it is closest to *B*. Find the position vector of *P*. [3]

Q3

The line l_1 passes through the point A, whose position vector is $3\mathbf{i}+7\mathbf{j}+3\mathbf{k}$, and is parallel to the vector $3\mathbf{i}+4\mathbf{j}+\mathbf{k}$. The line l_2 is given by the cartesian equation $x-2=\frac{3-y}{2}=\frac{z-5}{2}$.

The plane p_1 contains l_1 and is parallel to l_2. Another plane p_2 also contains l_1 and is perpendicular to p_1.

(i) Find the cartesian equation of p_1. [3]

(ii) Find the distance of l_2 to p_1. [2]

(iii) Find the equation of p_2 in the scalar product form. [2]

A particle P moves along a straight line c which lies in the plane p_2 and c passes through a point $(5,\frac{1}{2},-3)$. P hits the plane p_1 at A and rebounds to move along another straight line d in p_2. The angle between d and l_1 is the same as the angle between c and l_1.

(iv) Find the direction cosines of d. [6]

(v) Another particle, Q, is placed at the point $(\frac{25}{2},\frac{21}{2},-\frac{1}{2})$. Find the shortest distance PQ as P moves along d. [3]

Answers

Vectors Test 4

Q1

(i)

$\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = -\mathbf{a} - \mathbf{b} = -(\mathbf{a} + \mathbf{b})$

(ii)

Since ΔOCD is similar to ΔACB, OD parallel to AB.

$\frac{OD}{AB} = \frac{CO}{CA} = \frac{1}{2}$.

$\overrightarrow{OD} = \frac{1(-\mathbf{a}) + 1(\mathbf{b})}{2} = \frac{\mathbf{b} - \mathbf{a}}{2}$

(iii)

Let $\overrightarrow{OB} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix}$.

$$\overrightarrow{CB} = \begin{pmatrix} x \\ 0 \\ z \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} x-2 \\ 0 \\ z \end{pmatrix}$$

$$\overrightarrow{CB}.\overrightarrow{CO} = \begin{pmatrix} x-2 \\ 0 \\ z \end{pmatrix} . \begin{pmatrix} -2 \\ 0 \\ 0 \end{pmatrix} = \left|\overrightarrow{CB}\right|(2)\cos\frac{\pi}{6} ---(1)$$

$$\left|\overrightarrow{CB}\right| = 4\cos\frac{\pi}{6} = 2\sqrt{3}$$

$$-2x + 4 = 2\sqrt{3}(2)\frac{\sqrt{3}}{2}$$
$$x = -1$$

$$\left|\overrightarrow{OB}\right| = \left|\begin{pmatrix} -1 \\ 0 \\ z \end{pmatrix}\right| = 2$$

$$(-1)^2 + z^2 = 2^2$$
$$z^2 = 3$$
$$z = \pm\sqrt{3}$$

$$\overrightarrow{OB} = \begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix} \text{or} \begin{pmatrix} -1 \\ 0 \\ -\sqrt{3} \end{pmatrix} (\text{rejected} \because z\text{-component} > 0).$$

(iv)

Equation of line passing through *OB*:

$$\overrightarrow{OB} = \lambda\begin{pmatrix}-1\\0\\\sqrt{3}\end{pmatrix},\ \lambda \in \mathbb{R}$$

$$\frac{x-2}{3} = \mu \Rightarrow x = 2 + 3\mu$$

$$\frac{y}{3} = \mu \Rightarrow y = 3\mu$$

$$z - 1 = \mu \Rightarrow z = \mu + 1$$

Equation of line:

$$\mathbf{r} = \begin{pmatrix}2\\0\\1\end{pmatrix} + \mu\begin{pmatrix}3\\3\\1\end{pmatrix}, \mu \in \mathbb{R}$$

Direction vector of line is not parallel to direction vector of line passing through *O* and *B* since direction vectors of both lines are not scalar multiple of each other.

Solving equations simultaneously:

$$\begin{pmatrix}2+3\mu\\3\mu\\\mu+1\end{pmatrix} = \lambda\begin{pmatrix}-1\\0\\\sqrt{3}\end{pmatrix}$$

There is no value of λ and μ that satisfy the above equation.

Since the lines are not parallel and non-intersecting, the lines are skew.

Q2

(i)

$\overrightarrow{OD}=\begin{pmatrix}0\\0\\8\end{pmatrix}$, $\overrightarrow{OE}=\begin{pmatrix}7\\0\\5\end{pmatrix}$, $\overrightarrow{OF}=\begin{pmatrix}0\\7\\5\end{pmatrix}$. Hence,

$\overrightarrow{DE}=\begin{pmatrix}7\\0\\5\end{pmatrix}-\begin{pmatrix}0\\0\\8\end{pmatrix}=\begin{pmatrix}7\\0\\-3\end{pmatrix}$, $\overrightarrow{DF}=\begin{pmatrix}0\\7\\5\end{pmatrix}-\begin{pmatrix}0\\0\\8\end{pmatrix}=\begin{pmatrix}0\\7\\-3\end{pmatrix}$ and $\overrightarrow{EF}=\begin{pmatrix}0\\7\\5\end{pmatrix}-\begin{pmatrix}7\\0\\5\end{pmatrix}=\begin{pmatrix}-7\\7\\0\end{pmatrix}$

A vector perpendicular to the plane is

$=\overrightarrow{DE}\times\overrightarrow{DF}$

$=\begin{pmatrix}7\\0\\-3\end{pmatrix}\times\begin{pmatrix}0\\7\\-3\end{pmatrix}$

$=\begin{pmatrix}21\\21\\49\end{pmatrix}=7\begin{pmatrix}3\\3\\7\end{pmatrix}$

Cartesian equation of the plane is

$\begin{pmatrix}x\\y\\z\end{pmatrix}\bullet\begin{pmatrix}3\\3\\7\end{pmatrix}=\begin{pmatrix}0\\0\\8\end{pmatrix}\bullet\begin{pmatrix}3\\3\\7\end{pmatrix}\Rightarrow \mathbf{r}\bullet\begin{pmatrix}3\\3\\7\end{pmatrix}=56$

$\therefore 3x+3y+7z=56$

(ii)

Let the required angle be θ

$$\cos\theta = \frac{\left|\begin{pmatrix}3\\3\\7\end{pmatrix}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}\right|}{\sqrt{3^2+3^2+7^2}\sqrt{1}} = \frac{7}{\sqrt{67}}$$

$\theta \approx 31.2°$ (1 dec place)
(or 0.545 rad)

(iii)
Method 1

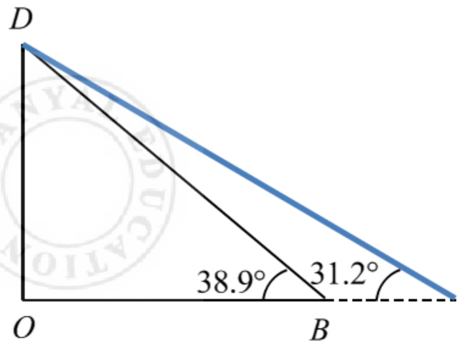
$\left|\overrightarrow{OB}\right| = \sqrt{7^2+7^2} = \sqrt{98}$

$\left|\overrightarrow{OD}\right| = 9$

Angle between DB and the ground
$= \angle OBD$

$= \tan^{-1}\left(\dfrac{8}{\sqrt{7^2+7^2}}\right)$

$\approx 38.9°$



From the diagram, the canvas will cover *B*.

Method 2

$\overrightarrow{OB} = \begin{pmatrix}7\\7\\0\end{pmatrix}$

Equation of perpendicular line passing through *B*, *l*:

$\mathbf{r} = \begin{pmatrix}7\\7\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\0\\1\end{pmatrix}, \lambda \in \mathbb{R}$

Using normal of plane to be $\begin{pmatrix}3\\3\\7\end{pmatrix}$ i.e. all entries are positive:

solve the equation of plane *DEF* and *l*:

$$\left[\begin{pmatrix}7\\7\\0\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}\right]\bullet\begin{pmatrix}3\\3\\7\end{pmatrix}=56$$

$\lambda=2$

Since $\lambda=2>0$, l and plane DEF intersect above the horizontal ground. So the canvas covers the point B.

Method 3

$$\begin{pmatrix}7\\7\\0\end{pmatrix}\bullet\begin{pmatrix}3\\3\\7\end{pmatrix}=42<56$$

Distance from O to plane parallel to DEF and passing through B is smaller than the distance between O and plane DEF. Hence B is beneath the canvas.

(iv)

Normal vector of the vertical wall is $\begin{pmatrix}1\\0\\0\end{pmatrix}$ and $(d,0,0)$ lies on the vertical wall.

$$\begin{pmatrix}d\\0\\0\end{pmatrix}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=d\Rightarrow \mathbf{r}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=d$$

Hence the equation of the vertical wall is $\mathbf{r}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=d$.

Direction vector of the line of intersection is

$$\begin{pmatrix}3\\3\\7\end{pmatrix}\times\begin{pmatrix}1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\7\\-3\end{pmatrix}$$

Let $(x,y,0)$ be the common point on lying on the two planes.

$$\begin{pmatrix}x\\y\\0\end{pmatrix}\bullet\begin{pmatrix}3\\3\\7\end{pmatrix}=56\Rightarrow 3x+3y=56$$

$$\begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = d \Rightarrow x = d$$

Solving the above equations simultaneously

$$3d + 3y = 56 \Rightarrow y = \frac{56 - 3d}{3}$$

$$\therefore \mathbf{r} = \begin{pmatrix} d \\ \frac{56-3d}{3} \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 7 \\ -3 \end{pmatrix}, \text{ where } \lambda \in \mathbb{R}.$$

(v)
For P to shine the brightest at point B, P must be as near as possible to B. Thus P is the foot of perpendicular from B to the roof.

Equation of the line passes through B and P:

$$\mathbf{r} = \begin{pmatrix} 7 \\ 7 \\ 0 \end{pmatrix} + \alpha \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix}, \alpha \in \mathbb{R}$$

Thus $\overrightarrow{OP} = \begin{pmatrix} 7+3\alpha \\ 7+3\alpha \\ 7\alpha \end{pmatrix}$ for some α.

Since P lies on the roof,

$$\begin{pmatrix} 7+3\alpha \\ 7+3\alpha \\ 7\alpha \end{pmatrix} \bullet \begin{pmatrix} 3 \\ 3 \\ 7 \end{pmatrix} = 56 \Rightarrow 42 + 67\alpha = 56$$

$$\therefore \alpha = \frac{14}{67}$$

Substitute $\alpha = \frac{14}{67}$ into $\overrightarrow{OP} = \begin{pmatrix} 7+3\alpha \\ 7+3\alpha \\ 7\alpha \end{pmatrix}$.

$$\overrightarrow{OP}=\begin{pmatrix}7+3\left(\frac{14}{67}\right)\\7+3\left(\frac{14}{67}\right)\\7\left(\frac{14}{67}\right)\end{pmatrix}=\begin{pmatrix}\frac{511}{67}\\\frac{511}{67}\\\frac{98}{67}\end{pmatrix}$$

Alternatively, use projection vector:

$$\overrightarrow{BD}=\begin{pmatrix}0\\0\\8\end{pmatrix}-\begin{pmatrix}7\\7\\0\end{pmatrix}=\begin{pmatrix}-7\\-7\\8\end{pmatrix}$$

To check for the direction of normal vector of *DEF*

$$\mathbf{n}=\begin{pmatrix}3\\3\\7\end{pmatrix}$$

$$\begin{pmatrix}3\\3\\7\end{pmatrix}.\overrightarrow{BD}=\begin{pmatrix}3\\3\\7\end{pmatrix}.\begin{pmatrix}-7\\-7\\8\end{pmatrix}=-21-21+56>0$$

Hence, angle between $\overrightarrow{BD}$ and **n** is acute.

$$\overrightarrow{BP}=\left(\overrightarrow{BD}.\mathbf{n}\right)\mathbf{n}$$

$$=\left(\begin{pmatrix}-7\\-7\\8\end{pmatrix}.\frac{\begin{pmatrix}3\\3\\7\end{pmatrix}}{\sqrt{9+9+49}}\right)\frac{\begin{pmatrix}3\\3\\7\end{pmatrix}}{\sqrt{9+9+49}}$$

$$=\frac{14}{67}\begin{pmatrix}3\\3\\7\end{pmatrix}$$

$$\overrightarrow{OP}=\overrightarrow{OB}+\overrightarrow{BP}$$

$$=\begin{pmatrix}7\\7\\0\end{pmatrix}+\frac{14}{67}\begin{pmatrix}3\\3\\7\end{pmatrix}$$

$$=\frac{1}{67}\begin{pmatrix}511\\511\\98\end{pmatrix}$$

Q3

(i) A vector equation of l_1 is $\mathbf{r}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\lambda\begin{pmatrix}3\\4\\1\end{pmatrix},\lambda\in\mathbb{R}$

Let $\mu=x-2=\dfrac{3-y}{2}=\dfrac{z-5}{2}$.

$\therefore x=2+\mu, y=3-2\mu, z=5+2\mu$

Then a vector equation of l_2 is $\mathbf{r}=\begin{pmatrix}2\\3\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\2\end{pmatrix},\lambda\in\mathbb{R}$

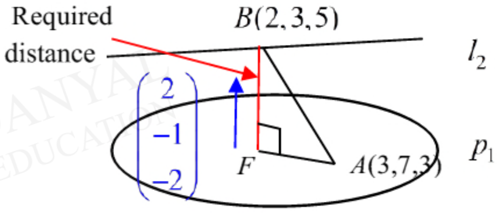
A vector perpendicular to p_1 is

$$\begin{pmatrix}3\\4\\1\end{pmatrix}\times\begin{pmatrix}1\\-2\\2\end{pmatrix}=\begin{pmatrix}10\\-5\\-10\end{pmatrix}=5\begin{pmatrix}2\\-1\\-2\end{pmatrix}//\begin{pmatrix}2\\-1\\-2\end{pmatrix}$$

Eqn of p_1: $\mathbf{r}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}3\\7\\3\end{pmatrix}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}=-7$

Cartesian eqn : $2x-y-2z=-7$

(ii) Distance of a line // to a plane is the distance between a point on this line to the plane



$$\text{Distance of } l_2 \text{ to } p_1 = BF = \frac{\left|\overrightarrow{AB}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right|}{\left|\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right|}=\frac{1}{3}\left|\begin{pmatrix}-1\\-4\\2\end{pmatrix}.\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right|=\frac{2}{3}$$

Alternative :

Equation of line BF: $\mathbf{r}=\begin{pmatrix}2\\3\\5\end{pmatrix}+\alpha\begin{pmatrix}2\\-1\\-2\end{pmatrix}$

$\overrightarrow{OF}=\begin{pmatrix}2\\3\\5\end{pmatrix}+\alpha\begin{pmatrix}2\\-1\\-2\end{pmatrix}$ for some $\alpha\in\mathbb{R}$

As F lies in p_1,

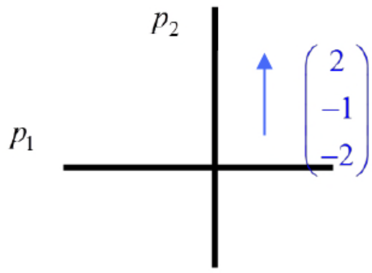
$$\left[\begin{pmatrix}2\\3\\5\end{pmatrix}+\alpha\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right].\begin{pmatrix}2\\-1\\-2\end{pmatrix}=-7$$

$$-9+9\alpha=-7$$

$$\therefore\alpha=\frac{2}{9}$$

$$\text{Distance of } l_2 \text{ to } p_1 = BF=\left|\overrightarrow{OF}-\overrightarrow{OB}\right| = \frac{2}{9}\left|\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right|=\frac{2}{9}\times3=\frac{2}{3}$$

(iii)



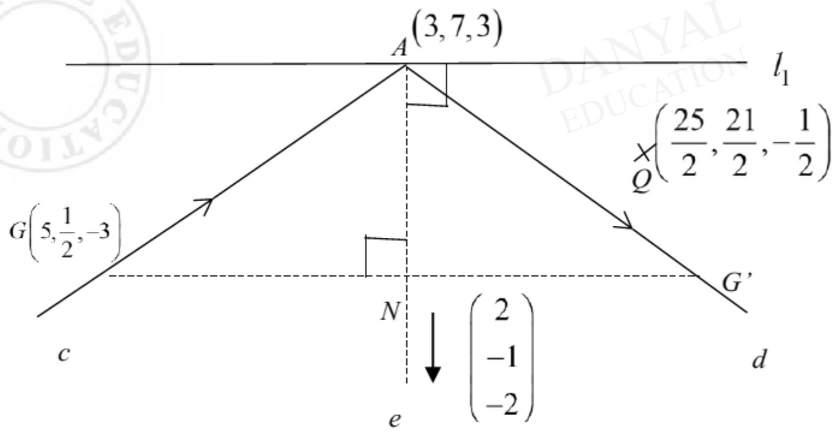
A vector perpendicular to p_2

$$=\begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}\times\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}=\begin{pmatrix} -7 \\ 8 \\ -11 \end{pmatrix}$$

Equation of p_2

$$\mathbf{r}.\begin{pmatrix} -7 \\ 8 \\ -11 \end{pmatrix}=\begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix}.\begin{pmatrix} -7 \\ 8 \\ -11 \end{pmatrix}=2$$

(iv)



line *d* is a reflection of line *c* in the line *e* which passes through *A*, is perpendicular to l_1 and p_1 and lying in p_2.

Eqn of line *e* : $\mathbf{r}=\begin{pmatrix} 3 \\ 7 \\ 3 \end{pmatrix}+\mu\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}$

Foot of perpendicular, *N*, from $\left(5,\frac{1}{2},-3\right)$ to line *e*

$$\left[\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}-\begin{pmatrix}5\\0.5\\-3\end{pmatrix}\right].\begin{pmatrix}2\\-1\\-2\end{pmatrix}=0$$

$$\left[\begin{pmatrix}-2\\6.5\\6\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}\right].\begin{pmatrix}2\\-1\\-2\end{pmatrix}=0$$

$$-4-6.5-12+\mu(4+1+4)=0$$

$$\mu=\frac{45}{18}=\frac{5}{2}$$

$$\overrightarrow{ON}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}8\\4.5\\-2\end{pmatrix}$$

$$\overrightarrow{ON}=\frac{1}{2}\left(\overrightarrow{OG}+\overrightarrow{OG'}\right)$$

$$\overrightarrow{OG'}=2\begin{pmatrix}8\\4.5\\-2\end{pmatrix}-\begin{pmatrix}5\\0.5\\-3\end{pmatrix}=\begin{pmatrix}11\\8.5\\-1\end{pmatrix}$$

$$\overrightarrow{AG'}=\begin{pmatrix}11\\8.5\\-1\end{pmatrix}-\begin{pmatrix}3\\7\\3\end{pmatrix}=\begin{pmatrix}8\\1.5\\-4\end{pmatrix}//\begin{pmatrix}16\\3\\-8\end{pmatrix}$$

Direction cosines of line d are $\frac{16}{\sqrt{329}},\frac{3}{\sqrt{329}}$ and $-\frac{8}{\sqrt{329}}$

or $-\frac{16}{\sqrt{329}},-\frac{3}{\sqrt{329}}$ and $\frac{8}{\sqrt{329}}$.

[**Alternative** to find $\overrightarrow{ON}$ - intersection of 2 lines]

Eqn of line e : $\mathbf{r}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}$

Eqn of line GN: $\mathbf{r}=\begin{pmatrix}5\\0.5\\-3\end{pmatrix}+\alpha\begin{pmatrix}3\\4\\1\end{pmatrix}$

At N,

$$\begin{pmatrix}3\\7\\3\end{pmatrix}+\mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}5\\0.5\\-3\end{pmatrix}+\alpha\begin{pmatrix}3\\4\\1\end{pmatrix}$$

$2\mu-3\alpha=2$

$\mu+4\alpha=6.5$

$2\mu+\alpha=6$

Solving, $\mu=2.5, \alpha=1$

$$\overrightarrow{ON}=\begin{pmatrix}3\\7\\3\end{pmatrix}+\frac{5}{2}\begin{pmatrix}2\\-1\\-2\end{pmatrix}=\begin{pmatrix}8\\4.5\\-2\end{pmatrix}$$

(v) Shortest distance from Q to line d

$$=\left|\frac{\overrightarrow{AQ}\times\begin{pmatrix}16\\3\\-8\end{pmatrix}}{\sqrt{329}}\right|$$

$$=\frac{1}{\sqrt{329}}\left|\left[\begin{pmatrix}\frac{25}{2}\\\frac{21}{2}\\-\frac{1}{2}\end{pmatrix}-\begin{pmatrix}3\\7\\3\end{pmatrix}\right]\times\begin{pmatrix}16\\3\\-8\end{pmatrix}\right|$$

$$=\frac{1}{\sqrt{329}}\left|\begin{pmatrix}\frac{19}{2}\\\frac{7}{2}\\-\frac{7}{2}\end{pmatrix}\times\begin{pmatrix}16\\3\\-8\end{pmatrix}\right|=\frac{1}{2\sqrt{329}}\left|\begin{pmatrix}19\\7\\-7\end{pmatrix}\times\begin{pmatrix}16\\3\\-8\end{pmatrix}\right|$$

$$=\frac{1}{2\sqrt{329}}\left|\begin{pmatrix}-35\\40\\-55\end{pmatrix}\right|=2.11\text{ (3 s.f.)}$$