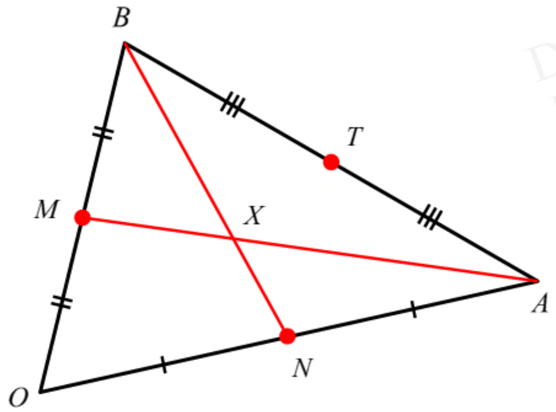
A Level H2 Math

Vectors Test 3

Q1

A median of a triangle is a line segment joining a vertex to the midpoint of the opposite side.

For the triangle shown below, O, A and B are vertices, where O is the origin, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The midpoints of OB, OA and AB are M, N and T respectively.



It is given that X is the point of intersection between the medians of triangle OAB from vertices A and B.

(i) Show that $\overrightarrow{OX} = \frac{1}{3}(\mathbf{a}+\mathbf{b})$. [4]

(ii) Prove that X also lies on OT, the median of triangle OAB from vertex O. [2]

The **centroid** of triangle OAB is the common point of intersection X between all three medians of the triangle.

Ray tracing is a technique in computer graphics rendering used to realistically capture the lighting effect in a scene being modelled. Starting from a chosen viewpoint, different rays are being traced backwards towards different parts of an object in the scene and reflected off the object. For each ray, if it reflects off the object and intersects a light source, then the part of the object at which the ray is reflected off would be made to appear brighter.

In a particular scene depicting a dolphin jumping out of the ocean, a ray is being traced back from a chosen viewpoint at V to the **centroid** X of a particular triangular facet defined by the vertices comprising the origin O, $A(5,\ 4,\ 6)$ and $B(-2,\ 2,\ 3)$, and then reflected off the facet at X, as shown in Figure 1.

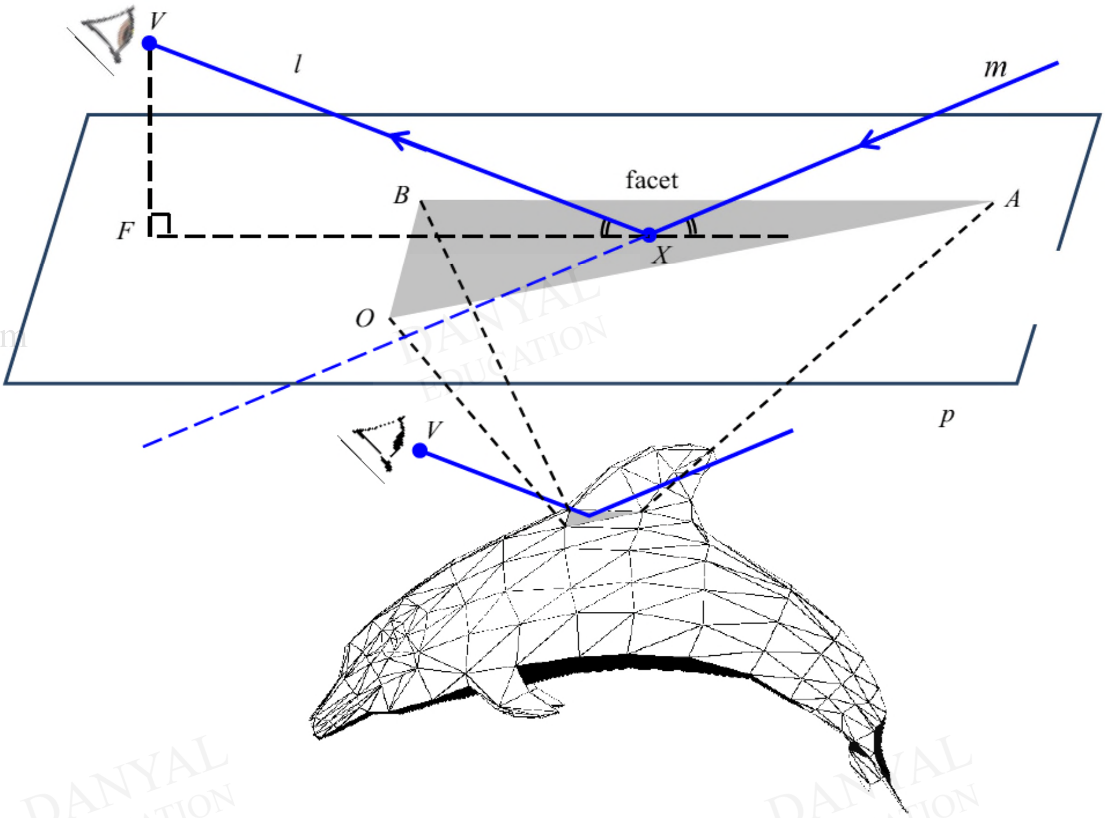


Figure 1

(iii) Show that the plane p which contains the triangular facet OAB can be represented by the cartesian equation $-3y+2z=0$. [2]

(iv) Given $V(1,-68,-37)$, determine the coordinates of the foot of perpendicular F from V to plane p. [4]

The reflected ray travels along a line m such that:

- both line VX (denoted by l) and line m lie in a plane that is perpendicular to plane p, and
- the angle between line l and plane p equals the angle between line m and plane p.

(v) By first finding two suitable points lying on line m, or otherwise, find a cartesian equation for line m. [5]

Q2

Referred to the origin O, the points A, B, P and Q have position vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{p}$ and $\mathbf{q}$ respectively, such that $|\mathbf{a}| = 2$, $\mathbf{b}$ is a unit vector, and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\frac{\pi}{4}$.

(i) Give a geometrical interpretation of $|\mathbf{b} \cdot \mathbf{a}|$. [1]

(ii) Find $|\mathbf{a} \times \mathbf{b}|$, leaving your answer in exact form. [2]

It is also given that $\mathbf{p} = 3\mathbf{a} + (\mu + 2)\mathbf{b}$ and $\mathbf{q} = (\mu + 3)\mathbf{a} + \mu\mathbf{b}$, where $\mu \in \mathbb{R}$.

(iii) Show that $\mathbf{p} \times \mathbf{q} = \left(\mu^2 + 2\mu + 6\right)(\mathbf{b} \times \mathbf{a})$. [3]

(iv) Hence find the smallest area of the triangle OPQ as μ varies. [3]

Q3

Given that $\mathbf{p} = 2\mathbf{i} + \alpha\mathbf{j} + \mathbf{k}$ and $\mathbf{q} = \alpha\mathbf{i} + \mathbf{j} + 6\mathbf{k}$, where α is a real constant and $\mathbf{w}$ is the position vector of a variable point W relative to the origin such that $\mathbf{w} \times \mathbf{p} = \mathbf{q}$.

(i) Find the value of α. [2]

(ii) Find the set of vectors $\mathbf{w}$ in the form $\{\mathbf{w} : \mathbf{w} = \mathbf{a} + \lambda\mathbf{b}, \lambda \in \mathbb{R}\}$. [3]

Answers

Vectors Test 3

Q1

(i) $\overrightarrow{AM} = -\mathbf{a} + \frac{1}{2}\mathbf{b}$

$\overrightarrow{BN} = -\mathbf{b} + \frac{1}{2}\mathbf{a}$

$\overrightarrow{OX} = \mathbf{a} + \overrightarrow{AX} \qquad = \mathbf{b} + \overrightarrow{BX}$

$= \mathbf{a} + \lambda(-\mathbf{a} + \frac{1}{2}\mathbf{b}) \quad = \mathbf{b} + \mu(-\mathbf{b} + \frac{1}{2}\mathbf{a})$ for some scalars λ, μ

$(1-\lambda)\mathbf{a} + \frac{\lambda}{2}\mathbf{b} \quad = \frac{\mu}{2}\mathbf{a} + (1-\mu)\mathbf{b}$

$\therefore \begin{cases} 1-\lambda = \frac{\mu}{2} \\ \frac{\lambda}{2} = 1-\mu \end{cases}$

Solving, $\lambda = \mu = \frac{2}{3}$

$\therefore\ \overrightarrow{OX} = \mathbf{a} + \frac{2}{3}(-\mathbf{a} + \frac{1}{2}\mathbf{b}) \quad = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ (shown)

Most students were able to obtain at least 2 out of 4 marks by using ratio theorem to find $\overrightarrow{AM}$ and $\overrightarrow{BN}$, but some were unsure how to continue.

Since **a** and **b** are non-parallel, we can compare the coefficients of the 2 vectors to obtain the 2 equations.

Method ①: (Using Equation of Lines)

$l_{AM}: \mathbf{r} = \mathbf{a} + \lambda\left(-\mathbf{a} + \frac{1}{2}\mathbf{b}\right),\ \lambda \in \mathbb{R}$

$l_{BN}: \mathbf{r} = \mathbf{b} + \mu\left(-\mathbf{b} + \frac{1}{2}\mathbf{a}\right),\ \mu \in \mathbb{R}$

Since X lies on both lines,

$\mathbf{a} + \lambda\left(-\mathbf{a} + \frac{1}{2}\mathbf{b}\right) = \mathbf{b} + \mu\left(-\mathbf{b} + \frac{1}{2}\mathbf{a}\right)$

$\begin{cases} 1-\lambda = \dfrac{\mu}{2} \\ \dfrac{\lambda}{2} = 1-\mu \end{cases}$

Solving, $\lambda = \mu = \frac{2}{3}$

$\therefore\ \overrightarrow{OX} = \mathbf{a} + \frac{2}{3}(-\mathbf{a} + \frac{1}{2}\mathbf{b}) \quad = \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$ (shown)

Method ②: (Using Ratio Theorem)

Using triangle OAM, $\overrightarrow{OX}=\dfrac{\lambda(\mathbf{a})+(1-\lambda)\left(\frac{1}{2}\mathbf{b}\right)}{\lambda+(1-\lambda)}=\lambda\mathbf{a}+\dfrac{(1-\lambda)}{2}\mathbf{b}$

Using triangle ONB, $\overrightarrow{OX}=\dfrac{\mu\left(\frac{1}{2}\mathbf{a}\right)+(1-\mu)\mathbf{b}}{\mu+(1-\mu)}=\dfrac{1}{2}\mu\mathbf{a}+(1-\mu)\mathbf{b}$

$$\lambda\mathbf{a}+\frac{(1-\lambda)}{2}\mathbf{b}=\frac{1}{2}\mu\mathbf{a}+(1-\mu)\mathbf{b}$$

$$\begin{cases}\lambda=\dfrac{\mu}{2}\\[2mm] \dfrac{1-\lambda}{2}=1-\mu\end{cases}$$

Solving, $\lambda=\frac{1}{3}$, $\mu=\frac{2}{3}$

$\overrightarrow{OX}=\frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}$ (shown)

(ii) $\overrightarrow{OT}=\frac{1}{2}(\mathbf{a}+\mathbf{b})$ using the midpoint theorem

$\overrightarrow{OX}=\frac{1}{3}(\mathbf{a}+\mathbf{b})$

$=\frac{2}{3}\left[\frac{1}{2}(\mathbf{a}+\mathbf{b})\right]=\frac{2}{3}\overrightarrow{OT}$

Since $\overrightarrow{OX}=k\overrightarrow{OT}$ for some scalar k where $0<k<1$, $\overrightarrow{OX}$ is parallel to $\overrightarrow{OT}$ with a common point O, hence X lies on OT.

Most students gave an incomplete proof for X lying on OT. It is essential to show that $\overrightarrow{OX}$ is a scalar multiple of $\overrightarrow{OT}$ and hence the 2 vectors are parallel.

(iii) $\overrightarrow{OA}=\begin{pmatrix}5\\4\\6\end{pmatrix}$, $\overrightarrow{OB}=\begin{pmatrix}-2\\2\\3\end{pmatrix}$

$$\overrightarrow{OA}\times\overrightarrow{OB}=\begin{pmatrix}5\\4\\6\end{pmatrix}\times\begin{pmatrix}-2\\2\\3\end{pmatrix}=\begin{pmatrix}(4)(3)-(6)(2)\\(6)(-2)-(5)(3)\\(5)(2)-(4)(-2)\end{pmatrix}$$

$$=\begin{pmatrix}0\\-27\\18\end{pmatrix}=9\begin{pmatrix}0\\-3\\2\end{pmatrix}$$

Since $\begin{pmatrix}0\\-3\\2\end{pmatrix}$ is perpendicular to the plane, and origin O is on the plane,

it is represented by $\mathbf{r}\cdot\begin{pmatrix}0\\-3\\2\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\cdot\begin{pmatrix}0\\-3\\2\end{pmatrix}=0.$

$\therefore\ -3y+2z=0$ (shown)

Most students were able to obtain the normal of the plane.

Since O is on the plane, the most direct method is to cross $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

(iv)

Line VF, $l_{VF}: \mathbf{r} = \begin{pmatrix} 1 \\ -68 \\ -37 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

Since F is on l_{VF}, $\overrightarrow{OF} = \begin{pmatrix} 1 \\ -68 \\ -37 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$, for some $\lambda \in \mathbb{R}$.

Since F is on p, $\overrightarrow{OF} \cdot \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = 0$.

$$\Rightarrow \left[\begin{pmatrix} 1 \\ -68 \\ -37 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} \right] \cdot \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = 0$$

$$130 + 13\lambda = 0$$

$$\lambda = -10$$

$$\overrightarrow{OF} = \begin{pmatrix} 1 \\ -68 \\ -37 \end{pmatrix} + (-10) \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -38 \\ -57 \end{pmatrix}$$

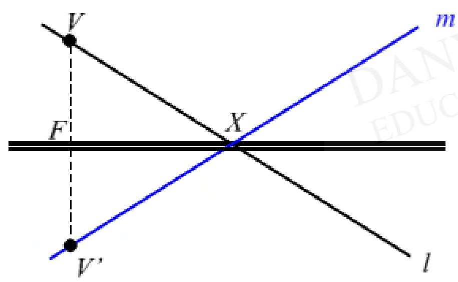
The coordinates of F is $(1, -38, -57)$.

Some students had the **misconception** that $\overrightarrow{VF} = \begin{pmatrix} 1 \\ -68 \\ -37 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix}$ and that since $\overrightarrow{VF}$ is parallel to the normal of plane, $\overrightarrow{VF} \cdot \begin{pmatrix} 0 \\ -3 \\ 2 \end{pmatrix} = 0$. The 2 vectors are parallel, **not** perpendicular.

A number of students were careless in solving for the value of λ.

(v)

$$\overrightarrow{OX} = \tfrac{1}{3}(\overrightarrow{OA} + \overrightarrow{OB}) = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$



Let V' be the reflection of V in plane p.

$\overrightarrow{OF} = \dfrac{\overrightarrow{OV} + \overrightarrow{OV'}}{2}$ $\quad \left[\text{or use } \overrightarrow{VF} = \overrightarrow{FV'}\right]$

$$\overrightarrow{OV'} = 2\overrightarrow{OF} - \overrightarrow{OV}$$

$$\overrightarrow{OV'} = 2\begin{pmatrix} 1 \\ -38 \\ -57 \end{pmatrix} - \begin{pmatrix} 1 \\ -68 \\ -37 \end{pmatrix} = \begin{pmatrix} 1 \\ -8 \\ -77 \end{pmatrix}$$

$$\overrightarrow{V'X} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} - \begin{pmatrix} 1 \\ -8 \\ -77 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \\ 80 \end{pmatrix} = 10 \begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix}$$

Line $m: \mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + k \begin{pmatrix} 0 \\ 1 \\ 8 \end{pmatrix}$, $k \in \mathbb{R}$

Line m: $x = 1,\ y - 2 = \dfrac{z-3}{8}$

Some students failed to notice that X is the centroid of triangle OAB although it was mentioned in the question.

Common mistakes include using $\overrightarrow{OX} = \dfrac{\overrightarrow{OV} + \overrightarrow{OV'}}{2}$ instead of $\overrightarrow{OF}$.

The above mistake could have been avoided if the student had **drawn a diagram**.

The question asked for a cartesian equation of line m, hence students were penalized for giving the vector equation form as the final answer.

Q2

(i)	Length of projection of $\mathbf{a}$ on to $\mathbf{b}$	Generally OK, but many gave the answer as length of projection of $\mathbf{b}$ onto $\mathbf{a}$.
(ii)	$\lvert\mathbf{a}\times\mathbf{b}\rvert = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert\sin\theta$ $= (2)(1)\sin\frac{\pi}{4}$ $= \sqrt{2}$	Many students mixed up the definition of dot and cross product, although $\sin\frac{\pi}{4}$ is the same as $\cos\frac{\pi}{4}$ which some students ended up with the correct final answer, but they still get penalized as they are using the wrong definition.
(iii)	$\mathbf{p}\times\mathbf{q}$ $= \left[3\mathbf{a}+(\mu+2)\mathbf{b}\right]\times\left[(\mu+3)\mathbf{a}+\mu\mathbf{b}\right]$ $= 3(\mu+3)(\mathbf{a}\times\mathbf{a})+3\mu(\mathbf{a}\times\mathbf{b})+(\mu^2+5\mu+6)(\mathbf{b}\times\mathbf{a})+\mu(\mu+2)(\mathbf{b}\times\mathbf{b})$ $= (-3\mu+\mu^2+5\mu+6)(\mathbf{b}\times\mathbf{a})$ $[\because \mathbf{a}\times\mathbf{a}=\mathbf{0}$ **and** $\mathbf{b}\times\mathbf{b}=\mathbf{0}]$ $= (\mu^2+2\mu+6)(\mathbf{b}\times\mathbf{a})$	Common mistakes: - $\mathbf{a}\times\mathbf{a}=\lvert\mathbf{a}\rvert^2$ - The third term in the expansion was $(\mu^2+5\mu+6)(\mathbf{a}\times\mathbf{b})$ instead of $(\mu^2+5\mu+6)(\mathbf{b}\times\mathbf{a})$; note that the direction of cross product is important. - $\underset{\sim}{a}\times\underset{\sim}{a} = \underset{\sim}{0}$ → null vector not $\underset{\sim}{a}\times\underset{\sim}{a} = 0$
(iv)	Area $OPQ = \frac{1}{2}\lvert(\mu^2+2\mu+6)\rvert\lvert(\mathbf{b}\times\mathbf{a})\rvert$ $=\frac{1}{2}\lvert(\mu^2+2\mu+6)\rvert\sqrt{2}$ $=\frac{\sqrt{2}}{2}\lvert(\mu+1)^2+5\rvert$	Common mistake: $\frac{1}{2}(\mu^2+2\mu+6)\mathbf{b}\times\mathbf{a}$ Note that the above expression is a vector, not magnitude.

Smallest Area $OPQ = \, = \frac{5\sqrt{2}}{2}$ unit^2

Q3

(i)

Method 1

$\mathbf{p.q}=\mathbf{0}$

$$\mathbf{p.q}=0$$

$$\begin{pmatrix}2\\\alpha\\1\end{pmatrix}.\begin{pmatrix}\alpha\\1\\6\end{pmatrix}=0$$

$$2\alpha+\alpha+6=0$$

$$\alpha=-2$$

Method 2 (for marking reference)

Let $\mathbf{w}=\begin{pmatrix}x\\y\\z\end{pmatrix}$.

$$\mathbf{w}\times\mathbf{p}=\mathbf{q}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}2\\\alpha\\1\end{pmatrix}=\begin{pmatrix}\alpha\\1\\6\end{pmatrix}$$

$$\begin{pmatrix}y-\alpha z\\2z-x\\\alpha x-2y\end{pmatrix}=\begin{pmatrix}\alpha\\1\\6\end{pmatrix}$$

Thus,

$y-\alpha z=\alpha$ ------(1)

$2z-x=1$ ------(2)

$\alpha x-2y=6$ ------(3)

$(2)\times\alpha+(3)$:

$2\alpha z-2y=\alpha+6$

$\Rightarrow 2(\alpha z-y)=\alpha+6$

$\Rightarrow 2(-\alpha)=\alpha+6$ (from (1))

$\Rightarrow \alpha=-2$

(ii)

Let $\mathbf{w}=\begin{pmatrix}x\\y\\z\end{pmatrix}$.

$$\mathbf{w}\times\mathbf{p}=\mathbf{q}$$

$$\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}-2\\1\\6\end{pmatrix}$$

$$\begin{pmatrix}y+2z\\2z-x\\-2x-2y\end{pmatrix}=\begin{pmatrix}-2\\1\\6\end{pmatrix}$$

$y+2z=-2$ ------(1)

$2z-x=1$ ------(2)

$-2x-2y=6$ ------(3)

Let $z=\lambda, \lambda\in\mathbb{R}$.

From (2): $x=-1+2\lambda$

From (1): $y+2\lambda=-2 \Rightarrow y=-2-2\lambda$

Thus, $\mathbf{w}=\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-1+2\lambda\\-2-2\lambda\\\lambda\end{pmatrix}=\begin{pmatrix}-1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix}$, $\lambda\in\mathbb{R}$, which is the vector equation of the straight line. The set of vectors is

$$\left\{\mathbf{w}:\mathbf{w}=\begin{pmatrix}-1\\-2\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\-2\\1\end{pmatrix}, \lambda\in\mathbb{R}\right\}$$