A Level H2 Math

Vectors Test 2

Q1

Relative to the origin O, the points A, B and C have position vectors $\mathbf{a}$, $\mathbf{a}+\mathbf{c}$ and $\mathbf{c}$ respectively. The point X is on AC produced such that $AC:CX$ is $2:3$ and the point Y is such that $AXYB$ is a parallelogram.

(i) The lines AY and BX intersect at the point N. Show that $\overrightarrow{ON}=\frac{1}{4}(7\mathbf{c}-\mathbf{a})$. [3]

(ii) Given that the area of triangle OAB is 4 square units, find the area of triangle OAN. [4]

(iii) Give a geometrical interpretation of $\left|\overrightarrow{OA}\cdot\frac{\overrightarrow{AN}}{\left|\overrightarrow{AN}\right|}\right|$. Use the results from part **(ii)** to show that

$\left|\overrightarrow{OA}\cdot\frac{\overrightarrow{AN}}{\left|\overrightarrow{AN}\right|}\right|=\frac{56}{|7\mathbf{c}-5\mathbf{a}|}$. [3]

* There was a typo error. Amend the dot product to a cross product for part (iii)

Q2

With reference to the origin O, the positon vectors of three points A, B and C are $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively. Given that $|\mathbf{a}|=4$, $|\mathbf{b}|=3$, $\mathbf{c}$ is a unit vector and the angle AOC is $\frac{\pi}{3}$ radians.

(i) Find the value of $\mathbf{a}\bullet\mathbf{c}$ and give the geometrical interpretation of this value. [2]

(ii) Given $\mathbf{a}-\mathbf{c}=k\mathbf{b}$ where $k\in\mathbb{R}$, $k\neq 0$. By considering $(\mathbf{a}-\mathbf{c})\bullet(\mathbf{a}-\mathbf{c})$, find the exact values of k. [3]

The point M divides OC in the ratio $OM:OC=2:3$.

(iii) Find the exact area of triangle AMC. [4]

Q3

The line l has equation $\dfrac{x-2}{4}=\dfrac{z+3}{1}$, $y=2$ and the plane p_1 has equation $\mathbf{r}.\begin{pmatrix}1\\1\\1\end{pmatrix}=16$.

Referred to the origin O, the position vector of the point A is $2\mathbf{i}+2\mathbf{j}-3\mathbf{k}$.

(i) Find the acute angle between the line l and the plane p_1. [2]

(ii) Find the coordinates of the foot of perpendicular, N, from point A to the plane p_1. [3]

(iii) Find the coordinates of the point B which is the reflection of A in plane p_1. [2]

(iv) Hence, determine the equation of the line which is a reflection of line l in the plane p_1. [4]

(v) Another plane, p_2, contains the point B and is parallel p_1. Determine the exact distance between p_1 and p_2. [2]

Answers

Vectors Test 2

Q1

(i)

$\overrightarrow{OA} = \mathbf{a},\ \overrightarrow{OB} = \mathbf{a}+\mathbf{c},\ \overrightarrow{OC} = \mathbf{c}$

$$\overrightarrow{OX} = \overrightarrow{OA} + \overrightarrow{AX}$$
$$= \overrightarrow{OA} + \frac{5}{2}\overrightarrow{AC}$$
$$= \mathbf{a} + \frac{5}{2}(\mathbf{c}-\mathbf{a})$$
$$= \frac{1}{2}(5\mathbf{c}-3\mathbf{a})$$

Alternatively:
By Ratio Theorem:

$$\overrightarrow{OC} = \frac{2\overrightarrow{OX}+3\overrightarrow{OA}}{5}$$
$$\overrightarrow{OX} = \frac{5\overrightarrow{OC}-3\overrightarrow{OA}}{2}$$
$$\overrightarrow{OX} = \frac{1}{2}(5\mathbf{c}-3\mathbf{a})$$

By midpoint theorem:

$$\overrightarrow{ON} = \frac{\overrightarrow{OB}+\overrightarrow{OX}}{2}$$
$$\overrightarrow{ON} = \frac{1}{2}\left[\mathbf{a}+\mathbf{c}+\frac{1}{2}(5\mathbf{c}-3\mathbf{a})\right]$$
$$= \frac{1}{4}(7\mathbf{c}-\mathbf{a})$$

(ii)

Area of triangle $OAB = \frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{OB}\right|$

$$4 = \frac{1}{2}\left|\mathbf{a}\times(\mathbf{a}+\mathbf{c})\right|$$
$$= \frac{1}{2}\left|\mathbf{a}\times\mathbf{c}\right| \qquad (\because \mathbf{a}\times\mathbf{a} = \underset{\sim}{0})$$
$$\Rightarrow \left|\mathbf{a}\times\mathbf{c}\right| = 8$$

Area of triangle $OAN = \frac{1}{2}\left|\overrightarrow{OA}\times\overrightarrow{ON}\right|$

$$= \frac{1}{2}\left|\mathbf{a}\times\frac{1}{4}(7\mathbf{c}-\mathbf{a})\right|$$
$$= \frac{7}{8}\left|\mathbf{a}\times\mathbf{c}\right| \qquad (\because \mathbf{a}\times\mathbf{a} = \underset{\sim}{0})$$
$$= \frac{7}{8}(8)$$
$$= 7 \quad \text{square units}$$

(iii) $\left|\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{|\overrightarrow{AN}|}\right|$ is the length of perpendicular from O to AN.

Alternative answer:

$\left|\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{|\overrightarrow{AN}|}\right|$ is the shortest distance from O to AN.

$\left|\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{|\overrightarrow{AN}|}\right|$ is the area of a parallelogram formed with vector $\overrightarrow{OA}$ and unit vector $\overrightarrow{AN}$ as its adjacent sides. **(Not recommended here)**

Area of triangle $OAN = 7$

$$\frac{1}{2}\left|\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{|\overrightarrow{AN}|}\right||\overrightarrow{AN}| = 7$$

$$\left|\overrightarrow{OA}\times\frac{\overrightarrow{AN}}{|\overrightarrow{AN}|}\right| = \frac{14}{|\overrightarrow{AN}|}$$

$$= \frac{14}{|\overrightarrow{ON}-\overrightarrow{OA}|}$$

$$= \frac{14}{\left|\frac{1}{4}(7\mathbf{c}-\mathbf{a})-\mathbf{a}\right|}$$

$$= \frac{56}{|7\mathbf{c}-5\mathbf{a}|}$$ **(shown)**

Q2

(i)

$$\mathbf{a} \cdot \mathbf{c} = 4(1)\cos\frac{\pi}{3} = 2$$

$|\mathbf{a} \cdot \mathbf{c}|$ is the length of projection of $\mathbf{a}$ onto $\mathbf{c}$

(ii)

$$(\mathbf{a}-\mathbf{c})\cdot(\mathbf{a}-\mathbf{c}) = k\mathbf{b}\cdot k\mathbf{b}$$

$$\mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{c} - \mathbf{c}\cdot\mathbf{a} + \mathbf{c}\cdot\mathbf{c} = k^2\mathbf{b}\cdot\mathbf{b}$$

$$|\mathbf{a}|^2 - 2\mathbf{a}\cdot\mathbf{c} + |\mathbf{c}|^2 = k^2|\mathbf{b}|^2$$

$$16 - 2(2) + 1 = 9k^2$$

$$k^2 = \frac{13}{9}$$

$$k = \pm\frac{\sqrt{13}}{3}$$

(iii)

$$\overrightarrow{MC} = \frac{1}{3}\mathbf{c}$$

Area of triangle AMC

$$= \frac{1}{2}\left|\overrightarrow{AC} \times \overrightarrow{MC}\right|$$

$$= \frac{1}{2}\left|(\mathbf{c}-\mathbf{a}) \times \frac{1}{3}\mathbf{c}\right|$$

$$= \frac{1}{6}|\mathbf{c}\times\mathbf{c} - \mathbf{a}\times\mathbf{c}|$$

$$= \frac{1}{6}|\mathbf{a}\times\mathbf{c}|$$

$$= \frac{1}{6}|\mathbf{a}||\mathbf{c}|\sin\left(\frac{\pi}{3}\right)$$

$$= \frac{1}{6}(4)(1)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{3}}{3}$$

Q3

$$l:\mathbf{r}=\begin{pmatrix}2\\2\\-3\end{pmatrix}+\lambda\begin{pmatrix}4\\0\\1\end{pmatrix},\ \lambda\in\mathbb{R}$$

Let θ be the angle between the line l and the plane p_1.

$$\sin\theta=\left|\frac{\begin{pmatrix}4\\0\\1\end{pmatrix}.\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{17}\sqrt{3}}\right|$$

$$=\left|\frac{5}{\sqrt{17}\sqrt{3}}\right|$$

$$\theta=44.4^\circ$$

(ii)

$$l_{AN}:\mathbf{r}=\begin{pmatrix}2\\2\\-3\end{pmatrix}+\mu\begin{pmatrix}1\\1\\1\end{pmatrix},\ \mu\in\mathbb{R}$$

$$\begin{pmatrix}2+\mu\\2+\mu\\-3+\mu\end{pmatrix}\bullet\begin{pmatrix}1\\1\\1\end{pmatrix}=16$$

$$2+\mu+2+\mu-3+\mu=16$$

$$3\mu=15$$

$$\mu=5$$

Coordinates of $N=(7,7,2)$

(iii)

Since N is the midpoint of A and B, using ratio theorem,

$$\overrightarrow{ON}=\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}$$

$$\overrightarrow{OB}=2\overrightarrow{ON}-\overrightarrow{OA}$$

$$=2\begin{pmatrix}7\\7\\2\end{pmatrix}-\begin{pmatrix}2\\2\\-3\end{pmatrix}=\begin{pmatrix}12\\12\\7\end{pmatrix}$$

Coordinates of $B=(12,12,7)$

Let C be the point of intersection of the line l and the plane p_1.

$$\begin{pmatrix} 2+4\lambda \\ 2 \\ -3+\lambda \end{pmatrix} \bullet \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 16$$

$$2+4\lambda+2-3+\lambda=16$$
$$5\lambda=15$$
$$\lambda=3$$

$$\overrightarrow{OC}=\begin{pmatrix} 14 \\ 2 \\ 0 \end{pmatrix}$$

$$\overrightarrow{BC}=\begin{pmatrix} 14 \\ 2 \\ 0 \end{pmatrix}-\begin{pmatrix} 12 \\ 12 \\ 7 \end{pmatrix}=\begin{pmatrix} 2 \\ -10 \\ -7 \end{pmatrix}$$

$$l_{BC}:\mathbf{r}=\begin{pmatrix} 14 \\ 2 \\ 0 \end{pmatrix}+s\begin{pmatrix} 2 \\ -10 \\ -7 \end{pmatrix},\ s\in\mathbb{R}$$

(v)

Since $AN=BN$,

$$BN=\sqrt{(2-7)^2+(2-7)^2+(-3-2)^2}$$
$$=\sqrt{(-5)^2+(-5)^2+(-5)^2}$$
$$=\sqrt{75}\text{units}$$
$$=5\sqrt{3}\text{units}$$