A Level H2 Math

Vectors Test 13

Q1

The point A has coordinates (3, 1, 1). The line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$, where λ is a parameter. P is a point on l when $\lambda = t$.

(i) Find cosine of the acute angle between AP and l in terms of t. Hence or otherwise, find the position vector of the point N on l such that N is the closest point to A. [6]

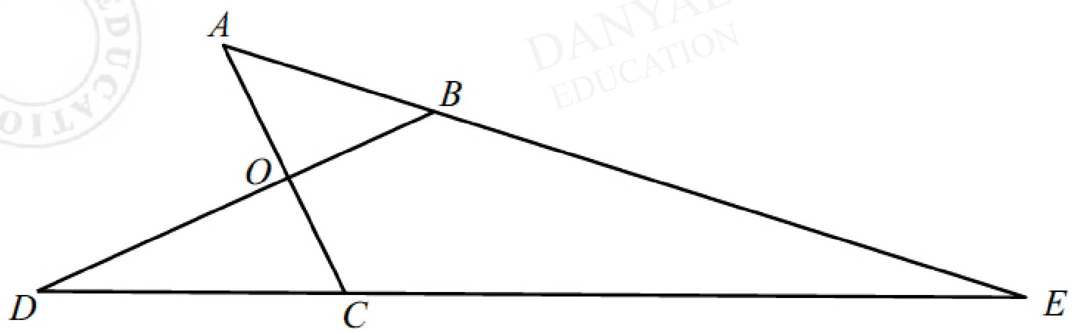
(ii) Find the coordinates of the point of reflection of A in l. [2]

The line L has equation $x = -1$, $2y = z + 2$.

(iii) Determine whether L and l are skew lines. [2]

(iv) Find the shortest distance from A to L. [3]

Q2



With reference to origin O, the points A, B, C and D are such that $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = -\mathbf{a}$ and $\overrightarrow{OD} = -2\mathbf{b}$. The lines AB and DC meet at E.

Find $\overrightarrow{OE}$ in terms of $\mathbf{a}$ and $\mathbf{b}$. [4]

Hence show that $\frac{BE}{AB} = 3$. [1]

It is given that A and E have coordinates (1, –4, 3) and (–3, 15, –5) respectively.

(i) Show that the lines AC and BD are perpendicular. [4]

(ii) Find the equation of the plane p that contains E and is perpendicular to the line BD. [2]

(iii) Find the distance between the line AC and p. [2]

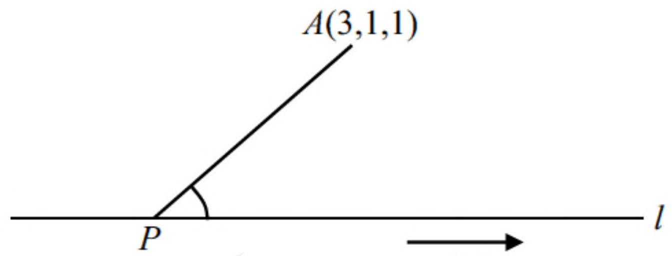
Q3

Referred to the origin O, the points A and B are such that $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. The point P on OA is such that $OP : PA = 2 : 3$, and the point Q on OB is such that $OQ : QB = 1 : 2$. Given that M is the mid-point of PQ, state the position vector of M in terms of $\mathbf{a}$ and $\mathbf{b}$.[1]
Show that the area of triangle OMP can be written as $k\,|\mathbf{a} \cdot \mathbf{b}|$, where k is a constant to be determined. [4]

Answers
Vectors Test 12

Q1

(i)



$\begin{pmatrix}2\\1\\1\end{pmatrix}$

P is a point on l with parameter t.

$$\Rightarrow \quad \overrightarrow{OP}=\begin{pmatrix}1\\1\\-1\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix},\ t\in\mathbb{R}$$

$$\overrightarrow{AP}=\begin{pmatrix}1\\1\\-1\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}-\begin{pmatrix}3\\1\\1\end{pmatrix}=\begin{pmatrix}-2\\0\\-2\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}$$

Let θ be the **acute** angle between BP and l.

Then,

$$\cos\theta=\frac{\left|\overrightarrow{AP}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}{\left|\overrightarrow{AP}\right|\left|\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}=\frac{\left|\left[\begin{pmatrix}-2\\0\\-2\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}\right|}{\sqrt{(-2+2t)^2+t^2+(-2+t)^2}\sqrt{4+1+1}}$$

$$=\frac{|(-4-2)+t(4+1+1)|}{\sqrt{4t^2-8t+4+t^2+t^2-4t+4}\sqrt{4+1+1}}$$

$$=\frac{6|t-1|}{\sqrt{6}\sqrt{6t^2-12t+8}}\quad\left(\text{or}\quad\frac{\sqrt{3}|t-1|}{\sqrt{3t^2-6t+4}}\right)$$

Need to read Qn carefully and do not make careless mistakes.

θ is acute, $\cos\theta>0$, so numerator needs to be positive.

$$\left[\begin{pmatrix}-2\\0\\-2\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}\right]\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}$$

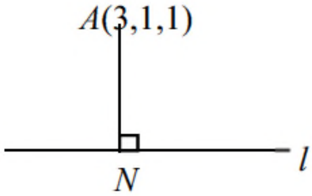
$$=\begin{pmatrix}-2\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}+t\begin{pmatrix}2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}2\\1\\1\end{pmatrix}$$

$$=(-4-2)+t(4+1+1)=-6+6t$$

Need to simplify the final answer, especially $\left|\overrightarrow{AP}\right|$

N is the closest point to A when $\theta = 90^\circ$.

$$\Rightarrow \cos 90^\circ = 0 = \frac{6|t-1|}{\sqrt{6}\sqrt{6t^2 - 12t + 8}}$$



$$\Rightarrow \quad t = 1$$

Thus, $\overrightarrow{ON} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + \begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}3\\2\\0\end{pmatrix}$

Alternative method

$\overrightarrow{ON} = \begin{pmatrix}1+2\lambda\\1+\lambda\\-1+\lambda\end{pmatrix}$, $\overrightarrow{AN} = \begin{pmatrix}2\lambda-2\\\lambda\\\lambda-2\end{pmatrix}$

$\overrightarrow{AN} \cdot \begin{pmatrix}2\\1\\1\end{pmatrix} = 0 \quad \Rightarrow \quad \lambda = 1$

$\therefore \overrightarrow{ON} = \begin{pmatrix}3\\2\\0\end{pmatrix}$

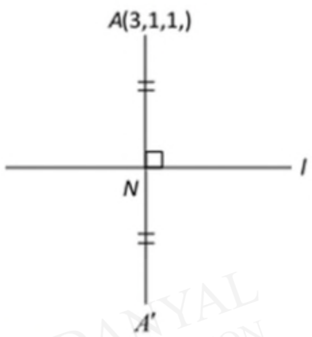
(ii) Let A' be the point of reflection of A in l.

Using ratio theorem,

$$\overrightarrow{ON} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OA'}\right)$$

$$\Rightarrow \quad \overrightarrow{OA'} = 2\overrightarrow{ON} - \overrightarrow{OA}$$

$$= 2\begin{pmatrix}3\\2\\0\end{pmatrix} - \begin{pmatrix}3\\1\\1\end{pmatrix} = \begin{pmatrix}3\\3\\-1\end{pmatrix}$$



Thus, the coordinate of A' are (3,3,−1).

Do not use long way to find point of reflection.

Eg. Begin with

$$\overrightarrow{BN} = \frac{1}{2}\left(\overrightarrow{BA} + \overrightarrow{BA'}\right)$$

Must answer the Qn.

Must write r

(iii) $l: \quad \underset{\sim}{r} = \begin{pmatrix}1\\1\\-1\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix}, \ t \in \mathbb{R}$

$L: \quad x = -1, \ 2y = z + 2 = \lambda$

i.e., $L: \quad \underset{\sim}{r} = \begin{pmatrix}-1\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}0\\\frac{1}{2}\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\-2\end{pmatrix} + m\begin{pmatrix}0\\1\\2\end{pmatrix}$

At point of intersection of lines l and L:

$$\begin{pmatrix}1\\1\\-1\end{pmatrix} + t\begin{pmatrix}2\\1\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\-2\end{pmatrix} + m\begin{pmatrix}0\\1\\2\end{pmatrix} \Rightarrow t = -1, \ m = 0$$

Since the point (−1,0,−2) lies on both l and L, the two lines **intersect** and thus cannot be skew lines. (Shown)

Cannot use the same parameter λ for both lines l and L.

You may also use this equation

for L: $\underset{\sim}{r} = \begin{pmatrix}-1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}0\\1\\2\end{pmatrix}$

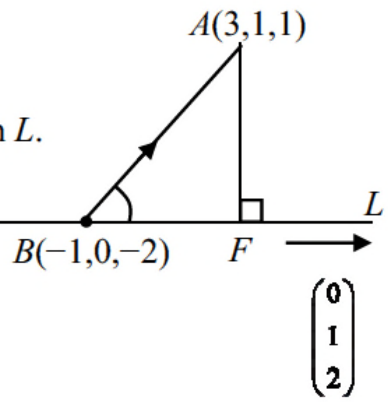
Need to know how to convert Cartesian equation to vector equation of a line.

Need to know how to use GC to solve the equation.

2 lines are not // and do not intersect $\Rightarrow$ they are skew lines

(iv) $L: \underset{\sim}{r} = \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} + m \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$

Let B be the point $(-1,0,-2)$ on L.



$$\overrightarrow{BA} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix}$$

Shortest distance from A to L

$$= \frac{\left| \overrightarrow{BA} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right|}{\left| \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right|} = \frac{1}{\sqrt{1+4}} \left| \begin{pmatrix} 4 \\ 1 \\ 3 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right|$$

$$= \frac{1}{\sqrt{5}} \left| \begin{pmatrix} -1 \\ -8 \\ 4 \end{pmatrix} \right| = \frac{\sqrt{1+64+16}}{\sqrt{5}} = \frac{9\sqrt{5}}{5}$$

Alternative method

Use × product not dot product

Don't divide by $\left| \overrightarrow{BA} \right|$

Marker's comments

This is a straight forward question, but many students still did not score it well. They either made careless mistakes or cannot remember the correct formulae.

For (i), many students can get $\overrightarrow{AP} = \begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$ correctly but copied it down wrongly when they use it to find cos θ.

Many students make the following mistakes:

- $\left[\begin{pmatrix} -2 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right] \cdot \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ -2 \end{pmatrix} + t\begin{pmatrix} 4 \\ 1 \\ 1 \end{pmatrix}$
- Drop | | in the numerator part half way in the calculation or totally did not put.
- Some students used $\overrightarrow{OP}$ instead of $\overrightarrow{AP}$ to find cos θ.
- Not many students use $\theta = 90^0$ to find $\overrightarrow{ON}$.

(ii) Many students forgot to give coordinates of *A'*.

(iii) Badly done for this part.

- Quite a number of students cannot obtain the correct vector equation of line *L*.
- Of those who had the correct equation at the point of intersection, many of them gave no solution for the equation. (Do not know how to use GC to solve?)
- For those who can get the intersection point, many students conclude that: "Since there are intersection point, therefore they are skew lines."

(iv) Badly done for this part.

Careless mistake: Used line *l* instead of line *L*.

Use wrong formula: for e.g., used dot product instead of cross product or divide by $|\overrightarrow{BA}|$

Q2

Equation of line AB is $\underset{\sim}{r}=\underset{\sim}{a}+\lambda(\underset{\sim}{b}-\underset{\sim}{a})$. Equation of line DC is $\underset{\sim}{r}=-\underset{\sim}{a}+\mu(-2\underset{\sim}{b}-(-\underset{\sim}{a}))$, i.e., $\underset{\sim}{r}=-\underset{\sim}{a}+\mu(-2\underset{\sim}{b}+\underset{\sim}{a})$. To find E, the point of intersection of lines AB and CD, consider $\underset{\sim}{a}+\lambda(\underset{\sim}{b}-\underset{\sim}{a})=-\underset{\sim}{a}+\mu(-2\underset{\sim}{b}+\underset{\sim}{a})$ $\Rightarrow\quad (1-\lambda)\underset{\sim}{a}+\lambda\underset{\sim}{b}=(-1+\mu)\underset{\sim}{a}-2\mu\underset{\sim}{b}$ $\Rightarrow\quad (2-\lambda-\mu)\underset{\sim}{a}=(-2\mu-\lambda)\underset{\sim}{b}$ Since $\underset{\sim}{a}$ is not parallel to $\underset{\sim}{b}$, $\begin{cases}2-\mu-\lambda=0 & \cdots(1)\\ -2\mu-\lambda=0 & \cdots(2)\end{cases}$ Solving (1) and (2), we have $\mu=-2$ and $\lambda=4$ $\therefore \overrightarrow{OE}=\underset{\sim}{a}+4(\underset{\sim}{b}-\underset{\sim}{a})=-3\underset{\sim}{a}+4\underset{\sim}{b}$ $\therefore \overrightarrow{BE}=\overrightarrow{OE}-\overrightarrow{OB}=-3\underset{\sim}{a}+4\underset{\sim}{b}-\underset{\sim}{b}=3(\underset{\sim}{b}-\underset{\sim}{a})=3\overrightarrow{AB}$ $\therefore \dfrac{BE}{AB}=3$	Students must know that there is no such things as $\dfrac{vector}{vector}$. In this case, students who have written $\dfrac{3(\mathbf{b}-\mathbf{a})}{(\mathbf{b}-\mathbf{a})}=3$ will not be given any credit.
(i) $\overrightarrow{OE}=-3\underset{\sim}{a}+4\underset{\sim}{b}=\begin{pmatrix}-3\\15\\-5\end{pmatrix}$ $\Rightarrow\quad -3\begin{pmatrix}1\\-4\\3\end{pmatrix}+4\underset{\sim}{b}=\begin{pmatrix}-3\\15\\-5\end{pmatrix}$ $\Rightarrow\quad 4\underset{\sim}{b}=\begin{pmatrix}-3\\15\\-5\end{pmatrix}+3\begin{pmatrix}1\\-4\\3\end{pmatrix}\quad\Rightarrow\quad \underset{\sim}{b}=\begin{pmatrix}0\\\frac{3}{4}\\1\end{pmatrix}$ $\underset{\sim}{a}\cdot\underset{\sim}{b}=\begin{pmatrix}0\\-4\\3\end{pmatrix}\cdot\begin{pmatrix}0\\\frac{3}{4}\\1\end{pmatrix}=-4\left(\frac{3}{4}\right)+3=0$ $\Rightarrow$ OA and OB are perpendicular $\Rightarrow$ AC and BD are perpendicular (as AC is parallel to OA and BD is parallel to OB)	Students must give clear explanation for every step. In this case, students must explain clearly why $OA\perp OB$ implies $AC\perp BD$.

(ii) Equation of the plane p is $\underset{\sim}{r}\cdot\begin{pmatrix}0\\3\\4\end{pmatrix}=\begin{pmatrix}-3\\15\\-5\end{pmatrix}\cdot\begin{pmatrix}0\\3\\4\end{pmatrix}$, i.e. $\underset{\sim}{r}\cdot\begin{pmatrix}0\\3\\4\end{pmatrix}=25$	
(iii) Distance between the line AC and the plane p = distance of O from p $=\dfrac{\begin{pmatrix}-3\\15\\-5\end{pmatrix}\cdot\begin{pmatrix}0\\3\\4\end{pmatrix}}{\sqrt{3^2+4^2}}=5$	

Marker's comments

The first part of this question is badly done. Students must know that problems involving vectors that are not given in the column vector way are very common in this syllabus. This question is just one example which requires you to find the intersection between two lines, in which position vectors of points on the lines are as generic vectors **a** and **b**. Students are advised to do more such practices from MSM and all other vectors revision resources that are given out.

Q3

$$\overrightarrow{OP}=\frac{2}{5}\mathbf{a} \qquad \overrightarrow{OQ}=\frac{1}{3}\mathbf{b}$$

$$\overrightarrow{OM}=\frac{1}{2}\left(\frac{2}{5}\mathbf{a}+\frac{1}{3}\mathbf{b}\right)$$

Area of triangle *OMP*

$$=\frac{1}{2}\left|\left(\frac{1}{2}\left(\frac{2}{5}\mathbf{a}+\frac{1}{3}\mathbf{b}\right)\right)\times\frac{2}{5}\mathbf{a}\right|$$

$$=\frac{1}{2}\left|\left(\left(\frac{1}{5}\mathbf{a}+\frac{1}{6}\mathbf{b}\right)\right)\times\frac{2}{5}\mathbf{a}\right|$$

$$=\frac{1}{2}\left|\frac{2}{25}\mathbf{a}\times\mathbf{a}+\frac{1}{15}\mathbf{b}\times\mathbf{a}\right|$$

$$=\frac{1}{2}\left|\frac{1}{15}\mathbf{b}\times\mathbf{a}\right|$$

$$=\frac{1}{30}\left|-\mathbf{a}\times\mathbf{b}\right|$$

$$=\frac{1}{30}\left|\mathbf{a}\times\mathbf{b}\right|$$