A Level H2 Math

Vectors Test 12

Q1

(a) The points A and B relative to the origin O have position vectors $3\mathbf{i}-\mathbf{j}+3\mathbf{k}$ and $-3\mathbf{i}+2\mathbf{j}$ respectively.

(i) Find the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$. [2]

(ii) Hence or otherwise, find the shortest distance from B to line OA. [2]

(b) The points C, D and E relative to the origin O have non-zero and non-parallel position vectors **c**, **d** and **e** respectively. Given that $(\mathbf{c}\times\mathbf{d})\cdot\mathbf{e}=0$, state with reason(s) the relationship between O, C, D and E. [2]

Q2

The plane p_1 has equation $\mathbf{r}=\begin{pmatrix}-1\\1\\16\end{pmatrix}+\lambda\begin{pmatrix}3\\-1\\2\end{pmatrix}+\mu\begin{pmatrix}2\\1\\1\end{pmatrix}$, where λ and μ are real parameters. The point A has position vector $5\mathbf{i}-6\mathbf{j}+7\mathbf{k}$.

(i) Find a cartesian equation of p_1. [3]

(ii) Find the position vector of the foot of perpendicular from A to p_1. [4]

The plane p_2 has equation $\mathbf{r}\cdot\begin{pmatrix}1\\-2\\5\end{pmatrix}=52$. The plane p_3 is obtained by reflecting p_2 about p_1. By considering the relationship between A and p_2, or otherwise, find a cartesian equation of p_3. [6]

Q3

(a) The vectors **a** and **b** are the position vector of points A and B respectively. It is given that $OA = 2\sqrt{7}$, $\mathbf{b} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{a} \cdot \mathbf{b} = -14$.

(i) Find angle AOB. [2]

(ii) State the geometrical meaning of $\left| \hat{\mathbf{a}} \cdot \mathbf{b} \right|$, where $\hat{\mathbf{a}}$ is the unit vector of **a**. [1]

(iii) Hence or otherwise, find the position vector of the foot of perpendicular from B to line OA in terms of **a**. [2]

(b) The non-zero vectors **p** and **q** are such that $\left| \mathbf{p} \times \mathbf{q} \right| = 2$. Given that **p** is a unit vector and $\mathbf{q} \cdot \mathbf{q} = 4$, show that **p** and **q** are perpendicular to each other. [3]

Answers
Vectors Test 12

Q1

(a)(i) Let θ be the angle between $\overrightarrow{OA}$ and $\overrightarrow{OB}$.

$$\cos\theta = \frac{\begin{pmatrix}3\\-1\\3\end{pmatrix}\bullet\begin{pmatrix}-3\\2\\0\end{pmatrix}}{\left|\begin{pmatrix}3\\-1\\3\end{pmatrix}\right|\left|\begin{pmatrix}-3\\2\\0\end{pmatrix}\right|}$$

$$\theta = \cos^{-1}\left(\frac{-11}{\sqrt{19}\sqrt{13}}\right) = 134.4° \;\; (1 \text{ d.p}) = 2.35 \text{ radian } (3 \text{ s.f})$$

(a)(ii) Let h be the shortest distance from B to line OA.

$$\sin 134.42° = \frac{h}{|\mathbf{b}|}$$

$$\begin{aligned} h &= \sqrt{13}\sin 134.42° \\ &= 2.5752 \\ &= 2.58 \text{ units } (3 \text{ s.f}) \end{aligned}$$

(b) Let $\mathbf{c}\times\mathbf{d} = \mathbf{s}$.

1) $\mathbf{s}\bullet\mathbf{e} = 0 \Rightarrow$ $\mathbf{s}$ is perpendicular to $\mathbf{e}$.

2) $\mathbf{c}\times\mathbf{d} = \mathbf{s} \Rightarrow$ $\mathbf{s}$ is perpendicular to both $\mathbf{c}$ and $\mathbf{d}$.

Since $\mathbf{s}$ is <u>perpendicular to **c**, **d** and **e**</u> and <u>**c**, **d** and **e** passes through common point O</u> $\Rightarrow$ points O, C, D and E are coplanar.

Q2

(i) $\begin{pmatrix}3\\-1\\2\end{pmatrix}\times\begin{pmatrix}2\\1\\1\end{pmatrix}=\begin{pmatrix}-3\\1\\5\end{pmatrix}$

$\mathbf{r}\cdot\begin{pmatrix}-3\\1\\5\end{pmatrix}=\begin{pmatrix}-1\\1\\16\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\5\end{pmatrix}=84$

Cartesian equation of p_1 is $-3x+y+5z=84$.

(ii) $\overrightarrow{OA}=\begin{pmatrix}5\\-6\\7\end{pmatrix}$

Let the foot of perpendicular from A to p_1 be F.

$\overrightarrow{OF}=\begin{pmatrix}5\\-6\\7\end{pmatrix}+\beta\begin{pmatrix}-3\\1\\5\end{pmatrix}=\begin{pmatrix}5-3\beta\\-6+\beta\\7+5\beta\end{pmatrix}$ for some $\beta\in\mathbb{R}$

Since F lies on p_1,

$\begin{pmatrix}5-3\beta\\-6+\beta\\7+5\beta\end{pmatrix}\cdot\begin{pmatrix}-3\\1\\5\end{pmatrix}=84$

$35\beta+14=84$

$\beta=2$

$\therefore\overrightarrow{OF}=\begin{pmatrix}5-6\\-6+2\\7+10\end{pmatrix}=\begin{pmatrix}-1\\-4\\17\end{pmatrix}$

Note that A lies on p_2 since $\begin{pmatrix}5\\-6\\7\end{pmatrix}\cdot\begin{pmatrix}1\\-2\\5\end{pmatrix}=52$.

Let A' be the point of reflection of A about p_1.

Note that A' lies on p_3.

$$\overrightarrow{OF}=\frac{\overrightarrow{OA}+\overrightarrow{OA'}}{2}$$

$$\overrightarrow{OA'}=2\overrightarrow{OF}-\overrightarrow{OA}=2\begin{pmatrix}-1\\-4\\17\end{pmatrix}-\begin{pmatrix}5\\-6\\7\end{pmatrix}=\begin{pmatrix}-7\\-2\\27\end{pmatrix}$$

$p_1 : -3x+y+5z=84$.

$p_2 : \mathbf{r}\cdot\begin{pmatrix}1\\-2\\5\end{pmatrix}=52 \Rightarrow x-2y+5z=52$

By GC, the line of intersection between p_1 and p_2 is $\mathbf{r}=\begin{pmatrix}-44\\-48\\0\end{pmatrix}+\alpha\begin{pmatrix}3\\4\\1\end{pmatrix}, \quad \alpha\in\mathbb{R}$

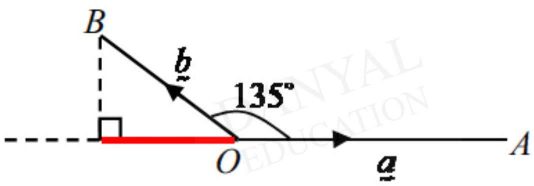
A vector parallel to p_3 is $\overrightarrow{OA'}-\begin{pmatrix}-44\\-48\\0\end{pmatrix}=\begin{pmatrix}-7\\-2\\27\end{pmatrix}-\begin{pmatrix}-44\\-48\\0\end{pmatrix}=\begin{pmatrix}37\\46\\27\end{pmatrix}$

$$\begin{pmatrix}37\\46\\27\end{pmatrix}\times\begin{pmatrix}3\\4\\1\end{pmatrix}=\begin{pmatrix}-62\\44\\10\end{pmatrix}=2\begin{pmatrix}-31\\22\\5\end{pmatrix}$$

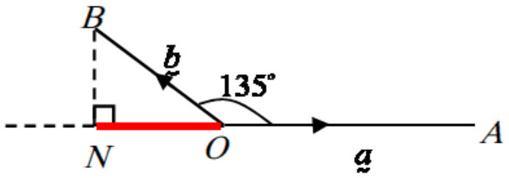
$$\mathbf{r}\cdot\begin{pmatrix}-31\\22\\5\end{pmatrix}=\begin{pmatrix}-7\\-2\\27\end{pmatrix}\cdot\begin{pmatrix}-31\\22\\5\end{pmatrix}=308$$

A cartesian equation of p_3 is $-31x+22y+5z=308$

Q3

(a)(i) <u>Given</u>: $\lvert\underset{\sim}{a}\rvert = 2\sqrt{7}$, $\underset{\sim}{b} = \begin{pmatrix}1\\2\\-3\end{pmatrix}$ and $\underset{\sim}{a}\cdot\underset{\sim}{b} = -14$ $\Rightarrow \lvert\underset{\sim}{a}\rvert\lvert\underset{\sim}{b}\rvert\cos AOB = -14$ $\Rightarrow \left(2\sqrt{7}\right)\sqrt{1+4+9}\cos AOB = -14$ $\Rightarrow \cos AOB = -\frac{7}{\sqrt{7}\sqrt{14}} = -\frac{1}{\sqrt{2}}$ $\Rightarrow AOB = 135^\circ$	Many students are confused between angles between two vectors and acute angles between 2 lines. Note that (1) If θ is the angle between two vectors **a**, **b**, then $\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$ (2) If θ is the acute angle between two lines with directional vectors **a**, **b**, then $\cos\theta = \frac{\lvert\mathbf{a}\cdot\mathbf{b}\rvert}{\lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert}$
(a)(ii) $\lvert\hat{\underset{\sim}{a}}\cdot\underset{\sim}{b}\rvert$ $= \lvert\lvert\hat{\underset{\sim}{a}}\rvert\lvert\underset{\sim}{b}\rvert\cos AOB\rvert$ $= \lvert\underset{\sim}{b}\rvert\lvert\cos AOB\rvert$  $\lvert\hat{\underset{\sim}{a}}\cdot\underset{\sim}{b}\rvert$ is the length of projection of $\underset{\sim}{b}$ on $\underset{\sim}{a}$.	

(a)(iii)



Let N be the foot of perpendicular from B to line OA.

Length of projection, $ON = \left|\hat{\underset{\sim}{a}} \cdot \underset{\sim}{b}\right| = \dfrac{\left|\underset{\sim}{a} \cdot \underset{\sim}{b}\right|}{\left|\underset{\sim}{a}\right|} = \dfrac{|-14|}{2\sqrt{7}} = \sqrt{7}$

Since $\angle AOB$ is an obtuse angle, $\overrightarrow{ON} = -\sqrt{7}\dfrac{\underset{\sim}{a}}{\left|\underset{\sim}{a}\right|} = -\frac{1}{2}\underset{\sim}{a}$

Alternative method to (a)(iii)

Let N be the foot of perpendicular from B to line OA.

Since N lies on line OA, $\overrightarrow{ON} = \lambda \underset{\sim}{a}$ for some $\lambda \in \mathbb{R}$.

Then, $\overrightarrow{BN} = \lambda \underset{\sim}{a} - \underset{\sim}{b}$

$\overrightarrow{BN} \perp \underset{\sim}{a}$

$\Rightarrow \quad \overrightarrow{BN} \cdot \underset{\sim}{a} = 0$

$\Rightarrow \quad \left(\lambda \underset{\sim}{a} - \underset{\sim}{b}\right) \cdot \underset{\sim}{a} = 0$

$\Rightarrow \quad \lambda \underset{\sim}{a} \cdot \underset{\sim}{a} = \underset{\sim}{a} \cdot \underset{\sim}{b}$

$\Rightarrow \quad \lambda \left|\underset{\sim}{a}\right|^2 = -14$

$\Rightarrow \quad \lambda \left(2\sqrt{7}\right)^2 = -14$

$\Rightarrow \quad \lambda = -\frac{1}{2}$

Thus, $\overrightarrow{ON} = -\frac{1}{2}\underset{\sim}{a}$

(b) <u>Given</u>: $\left|\underset{\sim}{p}\right| = 1$, $\quad \underset{\sim}{q} \cdot \underset{\sim}{q} = 4$

$\Rightarrow \left|\underset{\sim}{q}\right|^2 = 4 \Rightarrow \left|\underset{\sim}{q}\right| = 2$

<u>Given</u>: $\left|\underset{\sim}{p} \times \underset{\sim}{q}\right| = 2$

$\Rightarrow \quad \left|\underset{\sim}{p}\right|\left|\underset{\sim}{q}\right| \sin\theta = 2$, where θ is the angle between $\underset{\sim}{p}$ and $\underset{\sim}{q}$

$\Rightarrow \quad (1)(2)\sin\theta = 2$

$\Rightarrow \quad \sin\theta = 1$

$\Rightarrow \quad \theta = 90^\circ$

Thus, $\underset{\sim}{p}$ and $\underset{\sim}{q}$ are perpendicular to each other. (Shown)

Many students are not aware that $\left|\underset{\sim}{p} \times \underset{\sim}{q}\right| = \left|\underset{\sim}{p}\right|\left|\underset{\sim}{q}\right| \sin\theta$.

Note that we do not need to mod sinθ as θ denotes the angle between vectors which means that θ is between 0 and π. Hence, sinθ will always be +ve.

Marker's comments

Students must know that the definition for both dot and cross product (i.e. $|\underset{\sim}{p} \cdot \underset{\sim}{q}| = |\underset{\sim}{p}||\underset{\sim}{q}||\cos\theta|$ and $|\underset{\sim}{p} \times \underset{\sim}{q}| = |\underset{\sim}{p}||\underset{\sim}{q}|\sin\theta$) are very useful when solving problems that involve vectors which are not given in column vectors form.

Students who have applied using these definitions in this question had done well in this question.