A Level H2 Math

Vectors Test 11

Q1

Referred to the origin O, the points A, B and D are such that $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$ and $\overrightarrow{OD}=\mathbf{d}$. The point C is such that $OACB$ is a parallelogram and angle OAC is $\frac{2\pi}{3}$ radians.

(i) Given that $\mathbf{a}$ is a unit vector and $|\mathbf{b}|=4$, find the length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OA}$. [3]

(ii) Given that $\lambda\mathbf{a}+\mu\mathbf{b}+\mathbf{d}=\mathbf{0}$ and $\lambda+\mu+1=0$, show that A, B and D are collinear. [3]

If $\mu=4$, find the area of triangle OBD, leaving your answer in the form $k|\mathbf{a}\times\mathbf{b}|$, where k is a constant to be determined. [3]

Q2

With reference to an origin O, a particle P moves in space with position vector $(\lambda-\mu)\mathbf{i}+(1+2\mu)\mathbf{j}+(2-3\lambda)\mathbf{k}$. Another particle Q moves along the line l with equation $\mathbf{r}=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$, $t\in\mathbb{R}$.

(i) State the locus of P. [1]

(ii) Determine if the particles P and Q can meet. [3]

(iii) Find the shortest possible distance between P and Q. [2]

Another particle R moves along the line m with equation $\mathbf{r}=\begin{pmatrix}1\\k\\0\end{pmatrix}+s\begin{pmatrix}2\\-2\\-3k\end{pmatrix}$, $s\in\mathbb{R}$, where k is a constant.

(iv) Find condition(s) satisfied by k if lines l and m are skew lines. [3]

(v) A particle is shot from $X(0,-1,-5)$ perpendicularly toward the path of Q. Find the coordinates of the point where it crosses the path of Q. [2]

Q3

The position vectors of points A and B with respect to the origin O are $\mathbf{a}$ and $\mathbf{b}$ respectively where $\mathbf{a}$ and $\mathbf{b}$ are non-zero vectors. Point C lies on OA produced such that $4OA = AC$ and point D lies on OB produced such that $OB = BD$. The lines BC and AD meet at the point M.

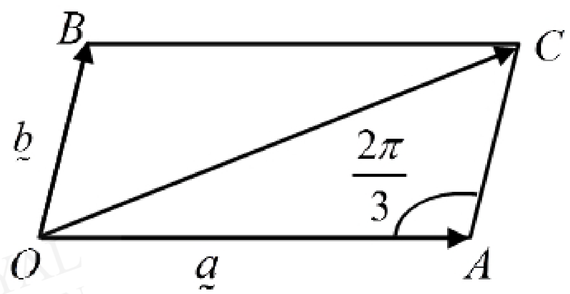
(i) Giving a necessary condition for $\mathbf{a}$ and $\mathbf{b}$, find the position vector of M in terms of $\mathbf{a}$ and $\mathbf{b}$. [5]

(ii) If $|\mathbf{a}| = 1$ and $|\mathbf{b}| = 2$, find the shortest distance of M from the line OC giving your answer in the form $k|\mathbf{a} \times \mathbf{b}|$ where k is a constant to be determined. [2]

Answers
Vectors Test 11

Q1

$\overrightarrow{OC} = \underset{\sim}{a} + \underset{\sim}{b}$



i	Length of projection of $\overrightarrow{OC}$ onto $\overrightarrow{OA}$ $= \left\|(\underset{\sim}{a}+\underset{\sim}{b})\cdot\hat{\underset{\sim}{a}}\right\|$ $= \left\|\underset{\sim}{a}\cdot\hat{\underset{\sim}{a}}+\underset{\sim}{b}\cdot\hat{\underset{\sim}{a}}\right\| = \left\|\underset{\sim}{a}\cdot\underset{\sim}{a}+\underset{\sim}{b}\cdot\underset{\sim}{a}\right\| \quad \because \underset{\sim}{a}=\hat{\underset{\sim}{a}}$ $= \left\|\left\|\underset{\sim}{a}\right\|^2+\left\|\underset{\sim}{b}\right\|\left\|\underset{\sim}{a}\right\|\cos\left(\pi-\frac{2\pi}{3}\right)\right\|$ $= \left\|1+4\left(\frac{1}{2}\right)\right\|$ $= 3$
ii	$\lambda\underset{\sim}{a}+\mu\underset{\sim}{b}+\underset{\sim}{d}=\underset{\sim}{0} \qquad ---(1)$ $\lambda+\mu+1=0 \qquad ---(2)$ Sub (2) into (1): $(-1-\mu)\underset{\sim}{a}+\mu\underset{\sim}{b}+\underset{\sim}{d}=\underset{\sim}{0}$ $\mu(\underset{\sim}{b}-\underset{\sim}{a})=\underset{\sim}{a}-\underset{\sim}{d}$ $\mu\overrightarrow{AB}=\overrightarrow{DA}$ Since $AB // DA$ and A is a common point, A, B and D are collinear
	Given $\mu=4$, $\underset{\sim}{d}=5\underset{\sim}{a}-4\underset{\sim}{b}$ Area of triangle OBD $=\frac{1}{2}\left\|\underset{\sim}{b}\times\underset{\sim}{d}\right\|$ $=\frac{1}{2}\left\|\underset{\sim}{b}\times(5\underset{\sim}{a}-4\underset{\sim}{b})\right\|$ $=\frac{1}{2}\left\|5\underset{\sim}{b}\times\underset{\sim}{a}-4\underset{\sim}{b}\times\underset{\sim}{b}\right\|$ $=\frac{5}{2}\left\|\underset{\sim}{b}\times\underset{\sim}{a}\right\| \qquad (\because \underset{\sim}{b}\times\underset{\sim}{b}=\underset{\sim}{0})$ $=\frac{5}{2}\left\|\underset{\sim}{a}\times\underset{\sim}{b}\right\|$ $\therefore k=\frac{5}{2}$

Q2

(i)

$$\overrightarrow{OP} = \begin{pmatrix} \lambda - \mu \\ 1 + 2\mu \\ 2 - 3\lambda \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$$

Locus of P is the plane with equation

$$\mathbf{r} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} + \mu \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}, \quad \lambda, \mu \in \mathbb{R}$$

(ii)

Normal of the locus of P (plane), $\begin{pmatrix} 1 \\ 0 \\ -3 \end{pmatrix} \times \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ -3 \end{pmatrix} = 0$$

Hence the line l and the plane are parallel.

Equation of the plane, $\mathbf{r} \cdot \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 3 \\ 2 \end{pmatrix} = 7$

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 6 \\ 2 \\ 3 \end{pmatrix} = 8 \neq 7$$

Hence l is parallel to the plane and does not lie in the plane.
Hence points P and Q will never meet.
[Note that it is not sufficient just to show that l is parallel to the plane as it may actually lie on it. One must still need to show that there is a point on l that is not on the plane]
Alternatively, one can check that

$$\left[\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}\right]\bullet\begin{pmatrix}6\\3\\2\end{pmatrix}\neq 7 \text{ for all } t\in\mathbb{R}.$$

(iii) Shortest distance between P and Q is the distance between the line and the parallel plane.

$$\frac{\left|\begin{pmatrix}1\\1\\0\end{pmatrix}\cdot\begin{pmatrix}6\\3\\2\end{pmatrix}-7\right|}{\left|\begin{pmatrix}6\\3\\2\end{pmatrix}\right|}=\frac{2}{7}$$

(iv) Lines l and m are non-parallel. Hence $k\neq 1$.

If the two lines intersect,

$$\begin{pmatrix}1\\k\\0\end{pmatrix}+s\begin{pmatrix}2\\-2\\-3k\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix} \quad \text{for some } s,t\in\mathbb{R}$$

$1+2s=1+2t$ ___(1)

$k-2s=1-2t$ ___(2)

$-3sk=-3t$ ___(3)

From (1), $s=t$

Substituting $s=t$ in (2), $k=1$

$s=t$ and $k=1$ satisfies (3)

Thus for the system of linear equations to be inconsistent, $k\neq 1$.

Hence lines l and m are skew when $k\neq 1$.

(v) Let F be the foot of perpendicular from X to line l.

$$\overrightarrow{OF}=\begin{pmatrix}1\\1\\0\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix} \text{ for some } t\in\mathbb{R}$$

$$\overrightarrow{XF}=\begin{pmatrix}1\\2\\5\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}$$

$$\left[\begin{pmatrix}1\\2\\5\end{pmatrix}+t\begin{pmatrix}2\\-2\\-3\end{pmatrix}\right]\cdot\begin{pmatrix}2\\-2\\-3\end{pmatrix}=0$$

$$-17+17t=0$$

$$t=1$$

$$\overrightarrow{OF}=\begin{pmatrix}1\\1\\0\end{pmatrix}+\begin{pmatrix}2\\-2\\-3\end{pmatrix}=\begin{pmatrix}3\\-1\\-3\end{pmatrix}$$

$F\equiv(3,-1,-3)$

Q3

(i)	Assume that $\mathbf{a}$ and $\mathbf{b}$ are non-parallel vectors. $\overrightarrow{OC} = 5\mathbf{a}$, $\overrightarrow{OD} = 2\mathbf{b}$ On the line BC, $\overrightarrow{OM} = \lambda(5\mathbf{a}) + (1-\lambda)\mathbf{b}$ On the line AD, $\overrightarrow{OM} = \mu(2\mathbf{b}) + (1-\mu)\mathbf{a}$ Since $\mathbf{a}$ and $\mathbf{b}$ are non-zero, non-parallel vectors, comparing coefficient $5\lambda = 1-\mu$ $2\mu = 1-\lambda$ $\Rightarrow$ $\lambda = \frac{1}{9}, \mu = \frac{4}{9}$ Thus $\overrightarrow{OM} = \frac{5}{9}\mathbf{a} + \frac{8}{9}\mathbf{b}$	Note that this is a condition on vectors, not points. You may use Ratio Theorem, or you may find direction vectors $\overrightarrow{BC}$ and $\overrightarrow{AD}$ to find the respective lines.
(ii)	Since $\mathbf{a}$ is a unit vector in the direction of OC, shortest distance $= \left\|\overrightarrow{OM} \times \mathbf{a}\right\|$ $= \left\|\left(\frac{5}{9}\mathbf{a} + \frac{8}{9}\mathbf{b}\right) \times \mathbf{a}\right\|$ $= \frac{8}{9}\left\|\mathbf{a} \times \mathbf{b}\right\|$ $k = \frac{8}{9}$	Some students mixed up the properties for cross product and dot product. $\mathbf{a} \times \mathbf{a} = \mathbf{0}$, $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$ Note that distance cannot be negative. Always check your algebraic workings when your answers appear counter-intuitive.