A Level H2 Math

Vectors Test 10

Q1

A computer-controlled machine can be programmed to make plane cuts by keying in the equation of the plane of the cut, and drill holes in a straight line through an object by keying in the equation of the drill line.

A 10cm × 20 cm ×30 cm cuboid is to be cut and drilled. The cuboid is positioned relative to the *x*-, *y*-and *z*-axes as shown in Figure 1.

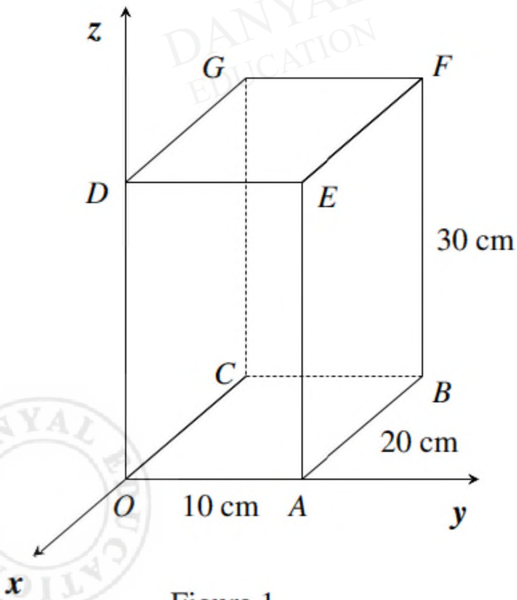


Figure 1

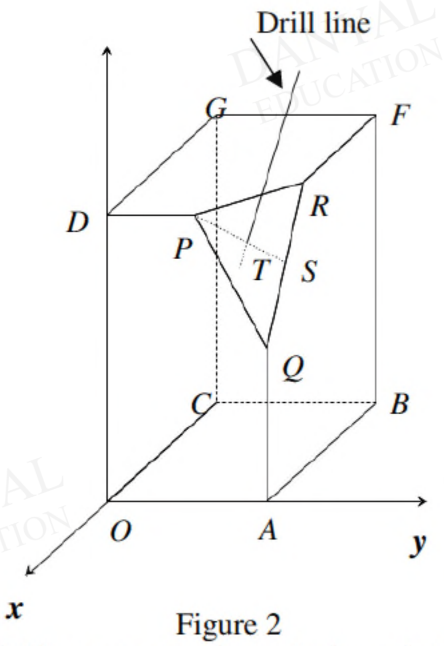


Figure 2

First, a plane cut is made to remove the corner at *E*. The cut goes through the points *P*, *Q* and *R* which are the midpoints of the sides *ED*, *EA* and *EF* respectively.

(i) Show that $\overrightarrow{PQ} = \begin{pmatrix} 0 \\ 5 \\ -15 \end{pmatrix}$ and $\overrightarrow{PR} = \begin{pmatrix} -10 \\ 5 \\ 0 \end{pmatrix}$. **[2]**

(ii) Find the cartesian equation of the plane, *p* that contains *P*, *Q* and *R*. **[2]**

(iii) Find the acute angle between *p* and the plane *DEFG*. **[2]**

A hole is then drilled perpendicular to triangle *PQR*, as shown in Figure 2. The hole passes through the triangle at the point *T* which divides the line *PS* in the ratio of 4:1, where *S* is the midpoint of *QR*.

(iv) Show that the point *T* has coordinates $(-4,\ 9,\ 24)$. **[3]**

(v) State the vector equation of the drill line. **[1]**

(vi) If the computer program continues drilling through the cuboid along the same line as in part **(v)**, determine the side of the cuboid that the drill exits from. Justify your answer. **[4]**

Q2

Referred to the origin O, the point A has position vector $-5\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and the point B has position vector $\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$. The plane π has equation:

$$\mathbf{r} = (1+\lambda-2\mu)\mathbf{i} + (3-2\lambda)\mathbf{j} + (\mu-2)\mathbf{k} \text{ where } \lambda, \mu \in \mathbb{R}$$

(i) Find the vector equation of plane π in scalar product form. [2]

(ii) Find the position vector of the foot of perpendicular, C, from A to π. [3]

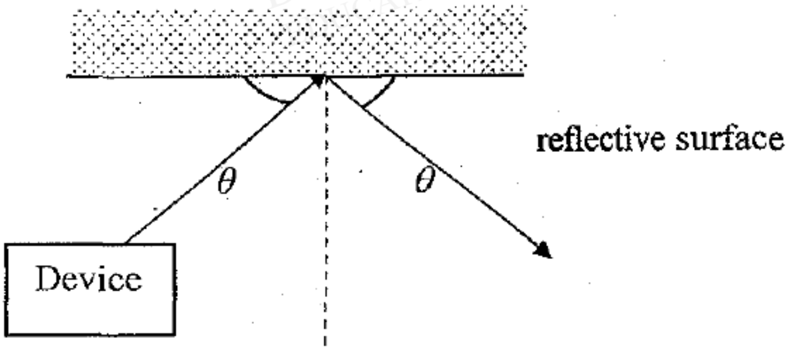
The line l_1 passes through the points A and B.

The line l_2 is the reflection of the line l_1 about the plane π. Find a vector equation of l_2. [3]

Q3

Physicists are investigating the reflective property of a particular reflective surface. The diagram below shows the set-up of a particular experiment, where a laser emitting device was placed at the point with coordinates (1, 2, 3). A laser beam was emitted in the direction parallel to $\mathbf{i}+\mathbf{k}$. The path of the emitted laser beam and its reflected path make the same angle θ with the reflective surface. The plane containing these two paths is perpendicular to the reflective surface.

Write down the vector equation of the path of the emitted laser beam. [1]



It is known that the reflective surface has equation $x+y+z=4$.

(i) Find θ. [3]

(ii) Show that the laser beam meets the reflective surface at the point $(0, 2, 2)$. [3]

(iii) Find the vector equation of the path of the reflected laser beam. [5]

Answers
Vectors Test 10

Q1

(i)

$\overrightarrow{OP}=\begin{pmatrix}0\\5\\30\end{pmatrix}$ $\overrightarrow{OQ}=\begin{pmatrix}0\\10\\15\end{pmatrix}$ $\overrightarrow{OR}=\begin{pmatrix}-10\\10\\30\end{pmatrix}$

$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\begin{pmatrix}0\\10\\15\end{pmatrix}-\begin{pmatrix}0\\5\\30\end{pmatrix}=\begin{pmatrix}0\\5\\-15\end{pmatrix}$ $\overrightarrow{PR}=\overrightarrow{OR}-\overrightarrow{OP}=\begin{pmatrix}-10\\10\\30\end{pmatrix}-\begin{pmatrix}0\\5\\30\end{pmatrix}=\begin{pmatrix}-10\\5\\0\end{pmatrix}$

(ii)

A normal to p

$$\begin{pmatrix}0\\1\\-3\end{pmatrix}\times\begin{pmatrix}-2\\1\\0\end{pmatrix}=\begin{pmatrix}3\\6\\2\end{pmatrix}$$

Equation of plane

$$\mathbf{r}\cdot\begin{pmatrix}3\\6\\2\end{pmatrix}=\begin{pmatrix}0\\10\\15\end{pmatrix}\cdot\begin{pmatrix}3\\6\\2\end{pmatrix}$$

$$\mathbf{r}\cdot\begin{pmatrix}3\\6\\2\end{pmatrix}=90$$

$$3x+6y+2z=90$$

Or any equivalent equation of plane

(iii)

A normal to the plane $EFGH=\begin{pmatrix}0\\0\\1\end{pmatrix}$

(or any equivalent vector)

$$\cos\theta=\frac{\left|\begin{pmatrix}0\\0\\1\end{pmatrix}\bullet\begin{pmatrix}3\\6\\2\end{pmatrix}\right|}{1\times\sqrt{9+36+4}}=\frac{|2|}{\sqrt{49}}$$

$$\theta=73.4^0$$

(iv)

$$\overrightarrow{OS}=\frac{1}{2}\left[\overrightarrow{OQ}+\overrightarrow{OR}\right]=\frac{1}{2}\left[\begin{pmatrix}0\\10\\15\end{pmatrix}+\begin{pmatrix}-10\\10\\30\end{pmatrix}\right]=\begin{pmatrix}-5\\10\\22\frac{1}{2}\end{pmatrix}$$

$$\overrightarrow{OT}=\frac{4\begin{pmatrix}-5\\10\\22\frac{1}{2}\end{pmatrix}+\begin{pmatrix}0\\5\\30\end{pmatrix}}{4+1}=\frac{1}{5}\begin{pmatrix}-20\\45\\120\end{pmatrix}=\begin{pmatrix}-4\\9\\24\end{pmatrix}$$

Hence the coordinates of T are $(-4,\ 9,\ 24)$.

(v)

Equation of the drill line: $\mathbf{r}=\begin{pmatrix}-4\\9\\24\end{pmatrix}+\lambda\begin{pmatrix}3\\6\\2\end{pmatrix},\ \lambda\in\mathbb{R}$.

(vi)
Shortlist the possible planes:
ODGC, *GCBF*, *OABC*

Equation of Plane *ODGC*: $\mathbf{r}\bullet\begin{pmatrix}0\\1\\0\end{pmatrix}=0$

Equation of Plane *OABC*: $\mathbf{r}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}=0$

Equation of Plane *GCBF*: $\mathbf{r}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=-20$

If the line of the drill exits from the cuboid, all of the following conditions must be satisfied:

$-20\le x\le 0;\ 0\le y\le 10;\ 0\le z\le 30$.
The intersection of plane *ODGC*

$$\begin{pmatrix}-4+3\lambda\\9+6\lambda\\24+2\lambda\end{pmatrix}\bullet\begin{pmatrix}0\\1\\0\end{pmatrix}=0$$

$9+6\lambda=0$

$\lambda=-\frac{3}{2}$

Position vector is $\begin{pmatrix}-4+3\left(-\frac{3}{2}\right)\\9+6\left(-\frac{3}{2}\right)\\24+2\left(-\frac{3}{2}\right)\end{pmatrix}=\begin{pmatrix}-\frac{17}{2}\\0\\21\end{pmatrix}$

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Hence the point of intersection has coordinates $\left(-\frac{17}{2},\ 0,\ 21\right)$.

Hence the drill line will exit from the side *ODGC*.

The intersection of plane *OABC*

$$\begin{pmatrix}-4+3\lambda\\9+6\lambda\\24+2\lambda\end{pmatrix}\bullet\begin{pmatrix}0\\0\\1\end{pmatrix}=0$$

$$24+2\lambda=0$$

$$\lambda=-12$$

Position vector is $\begin{pmatrix}-4+3(-12)\\9+6(-12)\\24+2(-12)\end{pmatrix}=\begin{pmatrix}-40\\-63\\0\end{pmatrix}$

Hence the point of intersection has coordinates $(-40,\ -63,\ 0)$.

Hence the drill line will not exit from the side *OABC*.

The intersection of plane *GCBF*

$$\mathbf{r}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=-20$$

$$\begin{pmatrix}-4+3\lambda\\9+6\lambda\\24+2\lambda\end{pmatrix}\bullet\begin{pmatrix}1\\0\\0\end{pmatrix}=-20$$

$$-4+3\lambda=-20$$

$$\lambda=-\frac{16}{3}$$

Position vector is $\begin{pmatrix}-4+3\left(-\frac{16}{3}\right)\\9+6\left(-\frac{16}{3}\right)\\24+2\left(-\frac{16}{3}\right)\end{pmatrix}=\begin{pmatrix}-20\\-23\\\frac{40}{3}\end{pmatrix}$

Hence the point of intersection has coordinates $\left(-20,\ -23,\ \frac{40}{3}\right)$.

Hence the drill line will not exit from the side *GCBF*.

Q2

(i)
Equation of plane is

$$\mathbf{r}=\begin{pmatrix}1\\3\\-2\end{pmatrix}+\lambda\begin{pmatrix}1\\-2\\0\end{pmatrix}+\mu\begin{pmatrix}-2\\0\\1\end{pmatrix},\ \lambda,\mu\in\mathbb{R}$$

A normal vector to plane is

$$\begin{pmatrix}1\\-2\\0\end{pmatrix}\times\begin{pmatrix}-2\\0\\1\end{pmatrix}=\begin{pmatrix}-2\\-1\\-4\end{pmatrix}$$

Hence vector equation of the plane is

$$\mathbf{r}\bullet\begin{pmatrix}2\\1\\4\end{pmatrix}=\begin{pmatrix}1\\3\\-2\end{pmatrix}\bullet\begin{pmatrix}2\\1\\4\end{pmatrix}$$

$$\mathbf{r}\bullet\begin{pmatrix}2\\1\\4\end{pmatrix}=-3$$

(ii)

$$l_{AC}:\quad \mathbf{r}=\begin{pmatrix}-5\\2\\2\end{pmatrix}+s\begin{pmatrix}2\\1\\4\end{pmatrix},\ s\in\mathbb{R}$$

Thus $\overrightarrow{OC}=\begin{pmatrix}-5\\2\\2\end{pmatrix}+s\begin{pmatrix}2\\1\\4\end{pmatrix}$ for some $s\in\mathbb{R}$.

Since C lies on the plane:

$$\left[\begin{pmatrix}-5\\2\\2\end{pmatrix}+s\begin{pmatrix}2\\1\\4\end{pmatrix}\right]\bullet\begin{pmatrix}2\\1\\4\end{pmatrix}=-3$$

$$2(-5+2s)+(2+s)+4(2+4s)=-3$$

$$s=-\frac{3}{21}$$

$$\text{Thus }\overrightarrow{OC}=\begin{pmatrix}2\left(-\frac{3}{21}\right)-5\\\left(-\frac{3}{21}\right)+2\\4\left(-\frac{3}{21}\right)+2\end{pmatrix}=\frac{1}{7}\begin{pmatrix}-37\\13\\10\end{pmatrix}$$

(iii)
Using mid-point theorem

$\overrightarrow{OA'} = 2\overrightarrow{OC} - \overrightarrow{OA}$

$$= \frac{2}{7}\begin{pmatrix} -37 \\ 13 \\ 10 \end{pmatrix} - \begin{pmatrix} -5 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} -39 \\ 12 \\ 6 \end{pmatrix}$$

B is the point of intersection of l_1 and π.

$\overrightarrow{BA'} = \overrightarrow{OA'} - \overrightarrow{OB}$

$$= \frac{1}{7}\begin{pmatrix} -39 \\ 12 \\ 6 \end{pmatrix} - \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}$$

$$= \frac{1}{7}\begin{pmatrix} -46 \\ -9 \\ 20 \end{pmatrix}$$

$$l_2: \quad \mathbf{r} = \frac{1}{7}\begin{pmatrix} -39 \\ 12 \\ 6 \end{pmatrix} + t\begin{pmatrix} -46 \\ -9 \\ 20 \end{pmatrix}, \ t \in \mathbb{R}$$ or

$$l_2: \quad \mathbf{r} = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix} + t\begin{pmatrix} -46 \\ -9 \\ 20 \end{pmatrix}, \ t \in \mathbb{R}$$

Q3

$$\underset{\sim}{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix}, \qquad \lambda \in \mathbb{R}$$

i

$$\cos\alpha = \frac{\begin{pmatrix}1\\0\\1\end{pmatrix}.\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{2}\sqrt{3}}$$

$$\alpha = 35.3^o$$
$$\theta = 90° - 35.3°$$
$$= 54.7°$$

ii Intersection of light beam with reflective surface:

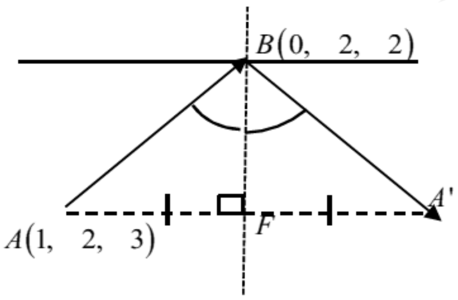
$$\left[\begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix}\right].\begin{pmatrix}1\\1\\1\end{pmatrix} = 4$$

$$6 + 2\lambda = 4$$

$$\lambda = -1$$

Coordinates of point of intersection = $(0, \quad 2, \quad 2)$.

iii Let F be the foot of perpendicular from device to normal line and A be the point $(1, \quad 2, \quad 3)$:



$$\overrightarrow{BF} = \left(\overrightarrow{BA}\cdot\hat{\underset{\sim}{n}}\right)\hat{\underset{\sim}{n}}$$

$$= \left[\frac{\begin{pmatrix}1-0\\2-2\\3-2\end{pmatrix}.\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{3}}\right]\frac{\begin{pmatrix}1\\1\\1\end{pmatrix}}{\sqrt{3}}$$

$$= \frac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

Using Ratio Theorem,

$$\overrightarrow{BF} = \frac{\overrightarrow{BA} + \overrightarrow{BA'}}{2}$$

$$\overrightarrow{BA'} = \frac{4}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} - \begin{pmatrix}1\\0\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1\\4\\1\end{pmatrix}$$

Equation of reflected light path:

$$r = \begin{pmatrix}0\\2\\2\end{pmatrix} + \alpha\begin{pmatrix}1\\4\\1\end{pmatrix}, \qquad \alpha \in \mathbb{R}$$