A Level H2 Math

Vectors Test 1

Q1

The points *O*, *A* and *B* are on a plane such that relative to the point *O*, the points *A* and *B* have non-parallel position vectors **a** and **b** respectively.
The point *C* with position vector **c** is on the plane *OAB* such that *OC* bisects the angle *AOB*.

Show that $\left(\frac{\mathbf{a}}{|\mathbf{a}|}-\frac{\mathbf{b}}{|\mathbf{b}|}\right)\cdot\mathbf{c}=0$. [2]

The lines *AB* and *OC* intersect at *P*. By first verifying that $\overrightarrow{OC}$ is parallel to $\frac{\mathbf{a}}{|\mathbf{a}|}+\frac{\mathbf{b}}{|\mathbf{b}|}$, show that the ratio of $AP:PB=|\mathbf{a}|:|\mathbf{b}|$. [6]

Q2

(a) **(i)** The unit vector **d** makes angles of $60°$ with both the *x*- and *y*-axes, and θ with the *z*-axis, where $0°\le\theta\le 90°$. Show that **d** is parallel to $\mathbf{i}+\mathbf{j}+\sqrt{2}\mathbf{k}$. [3]

(ii) The line *m* is parallel to **d** and passes through the point with coordinates $(2,-1,0)$. Find the coordinates of the point on *m* that is closest to the point with coordinates $(3,2,0)$. [3]

(b) The plane p_1 has equation $\mathbf{r}\cdot(\mathbf{i}+2\mathbf{j}+2\mathbf{k})=5$, and the line *l* has equation $\frac{x-a}{2}=\frac{y-1}{b}=-\frac{z}{2}$, where *a* and *b* are constants.

Given that *l* lies on p_1, show that $b=1$ and find the value of *a*. [2]

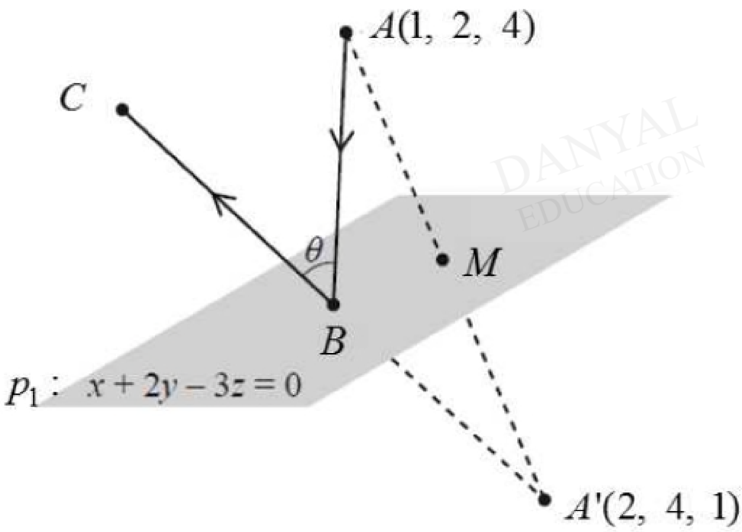
(i) The plane p_2 contains *l* and is perpendicular to p_1. Find the equation of p_2 in the form $\mathbf{r}\cdot\mathbf{n}=c$, where *c* is a constant to be determined. [3]

(ii) The variable point $P(x,y,z)$ is equidistant from p_1 and p_2. Find the cartesian equation(s) of the locus of *P*. [3]

Q3

Federal Aviation Administration data shows that there were an increase in aviation incidents caused by laser illuminations reported by pilots in 2015 and 2016. A simplified laboratory model is set up to investigate the effects of a laser beam on plexiglass, a common material used to make cockpit windscreen.

The piece of plexiglass is represented by a plane p_1 with equation $x+2y-3z=0$.



Referred to the origin, a laser beam ABC is fired from the point A with coordinates $(1,\ 2,\ 4)$, and is reflected at the point B on p_1 to form a reflected ray BC as shown in the diagram above. It is given that M is the midpoint of AA', where the point A' has coordinates $(2,\ 4,\ 1)$.

(i) Show that AA' is perpendicular to p_1. [2]

(ii) By finding the coordinates of M, show that M lies in p_1. [2]

The vector equation of the line AB is $\mathbf{r} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$, $\lambda \in \mathbb{R}$.

(iii) Find the coordinates of B. [2]

The acute angle between the incident ray AB and the reflected ray BC is θ (see diagram).

(iv) Given that $A'BC$ is a straight line, find the value of θ. Hence, or otherwise, write down the acute angle between the line AB and p_1. [3]

To reduce the effect of laser illumination on the pilot sitting in the cockpit at point A', a scientist proposes to include a protective film, represented by a plane p_2, such that the perpendicular distance from p_1 to p_2 is 0.5.

(v) State the possible cartesian equations of p_2. [2]

To further investigate the effects of a laser beam on plexiglass, separate laser beams are fired such that the incident ray AD is now a variable line which is also fired from the same point A and is reflected at the variable point D on p_1 to form a reflected ray DE.

(vi) Given that AD is perpendicular to the previous ray AB, find the minimum possible distance between B and D. [2]

(vii) Find the acute inclination of the reflected ray DE to the z-axis when DE is inclined at $60°$ to the x-axis and $45°$ to the y-axis. [3]

Answers

Vectors Test 1

Q1

$$\frac{\overrightarrow{OC}\cdot\overrightarrow{OA}}{|\overrightarrow{OC}||\overrightarrow{OA}|}=\frac{\overrightarrow{OC}\cdot\overrightarrow{OB}}{|\overrightarrow{OC}||\overrightarrow{OB}|}$$

$$\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|}=\frac{\mathbf{c}\cdot\mathbf{b}}{|\mathbf{b}|}\Rightarrow\frac{\mathbf{c}\cdot\mathbf{a}}{|\mathbf{a}|}-\frac{\mathbf{c}\cdot\mathbf{b}}{|\mathbf{b}|}=0\Rightarrow\mathbf{c}\cdot\left(\frac{\mathbf{a}}{|\mathbf{a}|}-\frac{\mathbf{b}}{|\mathbf{b}|}\right)=0$$

Alternatively

$$\left(\frac{\mathbf{a}}{|\mathbf{a}|}-\frac{\mathbf{b}}{|\mathbf{b}|}\right)\cdot\mathbf{c}=\frac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|}-\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|}$$

$$=\frac{|\mathbf{a}||\mathbf{c}|\cos\theta}{|\mathbf{a}|}-\frac{|\mathbf{b}||\mathbf{c}|\cos\theta}{|\mathbf{b}|}=0$$

This question was not well done with a significant number of students not attempting the question at all. Among those who attempted the questions, very few students managed to show that $AP:PB=|\mathbf{a}|:|\mathbf{b}|$.

Many students **wrongly assumed** that $|\mathbf{a}|=|\mathbf{b}|$.

$$\left(\frac{\mathbf{a}}{|\mathbf{a}|}+\frac{\mathbf{b}}{|\mathbf{b}|}\right)\cdot\left(\frac{\mathbf{a}}{|\mathbf{a}|}-\frac{\mathbf{b}}{|\mathbf{b}|}\right)=\left(\frac{\mathbf{a}\cdot\mathbf{a}}{|\mathbf{a}|^2}-\frac{\mathbf{b}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)$$

$$=\left(\frac{|\mathbf{a}|^2}{|\mathbf{a}|^2}-\frac{|\mathbf{b}|^2}{|\mathbf{b}|^2}\right)=1-1=0$$

P is on $l_{AB}\Rightarrow\overrightarrow{OP}=\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})=\lambda\mathbf{b}+(1-\lambda)\mathbf{a}$

P is on $l_{OC}\Rightarrow\overrightarrow{OP}=\mu\overrightarrow{OC}=\mu\left(\frac{\mathbf{a}}{|\mathbf{a}|}+\frac{\mathbf{b}}{|\mathbf{b}|}\right)$

Equating

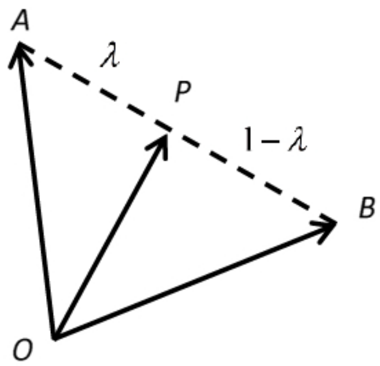
$$\lambda\mathbf{b}+(1-\lambda)\mathbf{a}=\mu\left(\frac{\mathbf{a}}{|\mathbf{a}|}+\frac{\mathbf{b}}{|\mathbf{b}|}\right)$$

Comparing coefficients of **a** and **b**

$\lambda=\frac{\mu}{|\mathbf{b}|}$ and $1-\lambda=\frac{\mu}{|\mathbf{a}|}$

Note that $AP:PB=\lambda:1-\lambda$, therefore

$AP:PB=\frac{\mu}{|\mathbf{b}|}:\frac{\mu}{|\mathbf{a}|}=|\mathbf{a}|:|\mathbf{b}|$.



Students need to know that for this question,

- ⇨ *OC* bisecting angle *AOB* **doesn't mean** that *AP*=*PB*.
- ⇨ $\overrightarrow{OP}$ & $\overrightarrow{OC}$ may **NOT** be perpendicular to $\overrightarrow{AB}$.
- ⇨ **c may not** be parallel to $\frac{\mathbf{a}}{|\mathbf{a}|}+\frac{\mathbf{b}}{|\mathbf{b}|}$ since $|\mathbf{a}|$ **may not** be equal to $|\mathbf{b}|$.
- ⇨ $\mathbf{a}+\mathbf{b}\neq\frac{\mathbf{a}}{|\mathbf{a}|}+\frac{\mathbf{b}}{|\mathbf{b}|}$
- ⇨ $\frac{\mathbf{a}\cdot\mathbf{a}}{|\mathbf{a}|^2}\neq\frac{\mathbf{a}^2}{|\mathbf{a}|^2}$

There was also poor usage of notation.
For example many students wrote "a" instead of "$\underset{\sim}{a}$" and also $\overrightarrow{AB}$ instead of $\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$.

Q2

(a)(i)	$\mathbf{d}=\cos 60°\mathbf{i}+\cos 60°\mathbf{j}+\cos\gamma\mathbf{k}$ $\cos^2 60°+\cos^2 60°+\cos^2\gamma=1$ $\Rightarrow \cos^2\gamma = 1-\frac{1}{4}-\frac{1}{4}=\frac{1}{2}$ $\Rightarrow \cos\gamma=\frac{1}{\sqrt{2}}\ (\because \gamma \text{ is acute})$ $\mathbf{d}=\frac{1}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}+\frac{1}{\sqrt{2}}\mathbf{k}\ //\ \mathbf{i}+\mathbf{j}+\sqrt{2}\mathbf{k}$	This part was rather poorly done, though most students can apply the geometrical definition of the scalar product and get 1 or 2 marks. Common errors include: (1) Not reading that **d** is a **unit** vector. (2) poor presentation with regard to the treatment of vectors and scalars, for e.g. **d** = 0.5. In addition, the <u>**show**</u>ing part needs to be worked on. Students have to present steps logically and quote relevant information from the question as part of their reasoning.
(a)(ii)	$m:\mathbf{r}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}$ $\left(\left(\begin{pmatrix}2\\-1\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}-\begin{pmatrix}3\\2\\0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}=0 \Rightarrow \left(\begin{pmatrix}-1\\-3\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}\right)\cdot\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}=0$ $\therefore (-1-3)+\lambda(1^2+1^2+\sqrt{2}^2)=0\Rightarrow\lambda=1$ Therefore position vector of point is $\begin{pmatrix}2\\-1\\0\end{pmatrix}+\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}=\begin{pmatrix}3\\0\\\sqrt{2}\end{pmatrix}$ Coordinates = $(3,0,\sqrt{2})$ <u>OR</u> $\overrightarrow{AN}=\left(\overrightarrow{AP}\cdot\mathbf{d}\right)\mathbf{d}=\frac{\left(\begin{pmatrix}3\\2\\0\end{pmatrix}-\begin{pmatrix}2\\-1\\0\end{pmatrix}\right)\cdot\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}}{\sqrt{1+1+2}}\frac{\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}}{\sqrt{1+1+2}}=\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}$ $\therefore\overrightarrow{ON}=\overrightarrow{OA}+\overrightarrow{AN}=\begin{pmatrix}2\\-1\\0\end{pmatrix}+\begin{pmatrix}1\\1\\\sqrt{2}\end{pmatrix}=\begin{pmatrix}3\\0\\\sqrt{2}\end{pmatrix}$ Coordinates = $(3,0,\sqrt{2})$	This is a simple part. No one should be getting this wrong. There were still students who upon not being able to show (a)(i), decided that (a)(ii) was not doable and had no attempt on it. A variety of methods were applied, though the easiest one is shown first on the left. Students who applied the vector of the projection with modulus sign instead of brackets could arrive at the answer as well, but they were not awarded the full marks due to a conceptual error. Of those who could do this part, around 50% of them lost the answer mark for not expressing in coordinates form.

4(b)	$l:\frac{x-a}{2}=\frac{y-1}{b}=-\frac{z}{2}\Rightarrow l:\mathbf{r}=\begin{pmatrix}a\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\b\\-2\end{pmatrix}$ $\begin{pmatrix}2\\b\\-2\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=0\Rightarrow 2+2b-4=0\Rightarrow b=1$ $\begin{pmatrix}a\\1\\0\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=5\Rightarrow a+2=5\Rightarrow a=3$	This was generally well-done, though a minority wrote $\begin{pmatrix}a+2\lambda\\1+b\lambda\\-2\lambda\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=5$ $\Rightarrow a+2\lambda+2+2b\lambda-4\lambda=5$ but obviously did not understand why $2\lambda+2b\lambda-4\lambda=0$.
(b)(i)	p_2 perpendicular to p_1 $\Rightarrow \mathbf{n}_1 // p_2$ $p_2:\mathbf{r}=\begin{pmatrix}3\\1\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}+\mu\begin{pmatrix}1\\2\\2\end{pmatrix}$ $\begin{pmatrix}2\\1\\-2\end{pmatrix}\times\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}6\\-6\\3\end{pmatrix}//\begin{pmatrix}2\\-2\\1\end{pmatrix}$ $p_2:\mathbf{r}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}3\\1\\0\end{pmatrix}\begin{pmatrix}2\\-2\\1\end{pmatrix}=4$	Some used longer method where they solved $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}2\\1\\-2\end{pmatrix}=0$ and $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=0$ Some remembered that the direction vector of line of intersection is $\mathbf{n}_1\times\mathbf{n}_2$ and wrote $\begin{pmatrix}x\\y\\z\end{pmatrix}\times\begin{pmatrix}1\\2\\2\end{pmatrix}=\begin{pmatrix}2\\1\\-2\end{pmatrix}$ but failed to include $\begin{pmatrix}x\\y\\z\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}=0$ as another condition. A significant minority made careless mistakes while computing the vector product. They should remind themselves how to check for correctness of the vector product.
(b)(ii)	$\frac{1}{\sqrt{9}}\left\|\begin{pmatrix}x-3\\y-1\\z\end{pmatrix}\cdot\begin{pmatrix}1\\2\\2\end{pmatrix}\right\|=\frac{1}{\sqrt{9}}\left\|\begin{pmatrix}x-3\\y-1\\z\end{pmatrix}\cdot\begin{pmatrix}2\\-2\\1\end{pmatrix}\right\|$ $\|x-3+2(y-1)+2z\|=\|2(x-3)-2(y-1)+z\|$ $\Rightarrow x+2y+2z-5=2x-2y+z-4$ $\Rightarrow x-4y-z=-1$ *or* $\Rightarrow x+2y+2z-5=-(2x-2y+z-4)$ $\Rightarrow 3x+3z=9\Rightarrow x+z=3$	Only less than 30 students are able to do this part. A handful gave good solutions, obtaining **n** as $\begin{pmatrix}1\\2\\2\end{pmatrix}+\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}3\\0\\3\end{pmatrix}$ or $\begin{pmatrix}1\\2\\2\end{pmatrix}-\begin{pmatrix}2\\-2\\1\end{pmatrix}=\begin{pmatrix}-1\\4\\1\end{pmatrix}$.

Q3

(i)

$$\overrightarrow{AA'} = \begin{pmatrix} 2-1 \\ 4-2 \\ 1-4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$$

Since $\overrightarrow{AA'} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = \underset{\sim}{n}_1$,

$\overrightarrow{AA'}$ is parallel to the normal of p_1,
and thus $\overrightarrow{AA'}$ is perpendicular to p_1.

Alternative Method:

Since $\overrightarrow{A'A} = \begin{pmatrix} 1-2 \\ 2-4 \\ 4-1 \end{pmatrix} = -\begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = -\underset{\sim}{n}_1$,

$\overrightarrow{A'A}$ is parallel to the normal of p_1,
and thus $\overrightarrow{A'A}$ is perpendicular to p_1

(ii)

Since M is the midpoint of A and A':

$$\overrightarrow{OM} = \frac{1}{2}\left[\begin{pmatrix} 2 \\ 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}\right] = \begin{pmatrix} 3/2 \\ 3 \\ 5/2 \end{pmatrix}$$

Coordinates of M are $\left(\frac{3}{2}, 3, \frac{5}{2}\right)$.

Note: Question asks for coordinates form.

Since $\frac{3}{2} + 2(3) - 3\left(\frac{5}{2}\right) = -6 + 6 = 0$,

M lies in p_1. (shown)

(iii)

$\overrightarrow{OB} = \begin{pmatrix} 1+\lambda \\ 2-\lambda \\ 4+2\lambda \end{pmatrix}$ for some $\lambda \in \mathbb{R}$.

Since B lies on p_1, $(1+\lambda) + 2(2-\lambda) - 3(4+2\lambda) = 0$

$$-7 - 7\lambda = 0$$
$$\lambda = -1$$

$\overrightarrow{OB} = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$ Coordinates of B are $(0, 3, 2)$.

Likewise for part **(vi)**.

(iv)

$\theta = \cos^{-1}\left|\dfrac{\overrightarrow{BA}\bullet\overrightarrow{A'B}}{\left|\overrightarrow{BA}\right|\left|\overrightarrow{A'B}\right|}\right|$

$= \cos^{-1}\left|\dfrac{\begin{pmatrix}1\\-1\\2\end{pmatrix}\bullet\begin{pmatrix}-2\\-1\\1\end{pmatrix}}{\sqrt{6}\sqrt{6}}\right|$

$= \cos^{-1}\left|\dfrac{1}{6}\right|$

$= 80.4°$ (1 d.p.)

Note:
You are expected to recognize that $\overrightarrow{A'B} = \overrightarrow{BC}$.

Hence, acute angle between the line AB and p_1

$= \dfrac{180° - 80.4°}{2}$

$= 49.8°$ (1 d.p.)

(v)

Possible cartesian equations of p_2:

$x + 2y - 3z = -\dfrac{\sqrt{14}}{2}$ or $x + 2y - 3z = \dfrac{\sqrt{14}}{2}$

(vi)

As incident ray AD varies, D is nearest to origin when OD is the shortest. Note that p_1 contains the origin.

$AB = \left|\begin{pmatrix}-1\\1\\-2\end{pmatrix}\right| = \sqrt{6}$

$\cos 49.8° = \dfrac{\sqrt{6}}{BD}$

$\Rightarrow BD = \dfrac{\sqrt{6}}{\cos 49.8°} = 3.79$ units (3 s.f.)

(vii)

Let γ be the required angle of inclination:

$\cos^2 60° + \cos^2 45° + \cos^2 \gamma = 1$

$\dfrac{1}{4} + \dfrac{1}{2} + \cos^2 \gamma = 1$

$\cos \gamma = \pm\dfrac{1}{2}$

$\therefore \gamma = 60°$ (since γ is acute)