A Level H2 Math

Sigma Notation Test 6

Q1

(i) Prove by the method of differences that

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \frac{1}{4} - \frac{k}{2(n+1)(n+2)},$$

where k is a constant to be determined. [5]

(ii) Explain why $\sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)}$ is a convergent series, and state its value. [2]

(iii) Using your answer in part **(i)**, show that $\sum_{r=1}^{n} \frac{1}{(r+2)^3} < \frac{1}{4}$. [2]

Q2

Given that $\sin\left[(n+1)x\right] - \sin\left[(n-1)x\right] = 2\cos nx \sin x$, show that

$$\sum_{r=1}^{n} \cos rx = \frac{\sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\ .$$ [4]

Hence express

$$\cos^2\left(\frac{x}{2}\right) + \cos^2(x) + \cos^2\left(\frac{3x}{2}\right) + \ldots + \cos^2\left(\frac{11x}{2}\right)$$

in the form $a\left(\frac{\sin bx}{\sin cx} + d\right)$, where a, b, c and d are real numbers. [3]

Q3

(i) Show that if $a_r = T_r - T_{r-1}$ for $r = 1, 2, 3, \ldots$, and $T_0 = 0$, then

$$\sum_{r=1}^{n} a_r = T_n.$$ [1]

(ii) Deduce that $\sum_{r=1}^{n} \pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right] = n^2\pi^{-n}$. [3]

(iii) Hence, find the exact value of $\sum_{r=4}^{20} \pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$. [2]

Answers

Sigma Notation Test 6

Q1

(i) Let $\dfrac{1}{r(r+1)(r+2)} = \dfrac{A}{r} + \dfrac{B}{r+1} + \dfrac{C}{r+2}$

Using 'cover-up' rule,

$A = \dfrac{1}{2}, \qquad B = -1, \qquad C = \dfrac{1}{2}$

$\therefore \dfrac{1}{r(r+2)} = \dfrac{1}{2r} - \dfrac{1}{r+1} + \dfrac{1}{2(r+2)}$

$$\sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{n}\left(\frac{1}{2r} - \frac{1}{r+1} + \frac{1}{2(r+2)}\right)$$

$$\begin{aligned} = \Bigg[& \frac{1}{2} - \frac{1}{2} + \frac{1}{6} \\ & + \frac{1}{4} - \frac{1}{3} + \frac{1}{8} \\ & + \frac{1}{6} - \frac{1}{4} + \frac{1}{10} \\ & + \frac{1}{8} - \frac{1}{5} + \frac{1}{12} \\ & + \ldots \\ & + \frac{1}{2(n-2)} - \frac{1}{n-1} + \frac{1}{2n} \\ & + \left(\frac{1}{2n-1}\right) - \frac{1}{n} + \frac{1}{2(n+1)} \\ & + \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)} \Bigg] \end{aligned}$$

$$= \frac{1}{2} - \frac{1}{2} + \frac{1}{4} + \frac{1}{2(n+1)} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

$$= \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

$$= \frac{1}{4} - \frac{1}{2(n+1)(n+2)} \text{ (proven)}$$

$\therefore k = 1$

(ii)

$$\text{As } n \to \infty, \frac{1}{2(n+1)(n+2)} \to 0, \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)} \to \frac{1}{4}$$

$$\therefore \sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} \text{ is a convergent series.}$$

$$\therefore \sum_{r=1}^{\infty} \frac{1}{r(r+1)(r+2)} = \frac{1}{4}$$

(iii)

For all $r \geq 1$,

$$(r+2)^3 > r(r+1)(r+2)$$

$$\frac{1}{(r+2)^3} < \frac{1}{r(r+1)(r+2)}$$

$$\sum_{r=1}^{n} \frac{1}{(r+2)^3} < \sum_{r=1}^{n} \frac{1}{r(r+1)(r+2)}$$

$$\sum_{r=1}^{n} \frac{1}{(r+2)^3} < \frac{1}{4} - \frac{1}{2(n+1)(n+2)}$$

$$\sum_{r=1}^{n} \frac{1}{(r+2)^3} < \frac{1}{4} \qquad \left(\because \frac{1}{2(n+1)(n+2)} > 0 \text{ for all } n \geq 1 \right)$$

Q2

Given: $2\cos nx \sin x = \sin(n+1)x - \sin(n-1)x$

Thus,

$$2\cos x \ \sin x = \sin 2x - \sin 0x$$
$$2\cos 2x \sin x = \sin 3x - \sin x$$
$$2\cos 3x \sin x = \sin 4x - \sin 2x$$
$$\ldots$$
$$2\cos(n-2)x \sin x = \sin(n-1)x - \sin(n-3)x$$
$$2\cos(n-1)x \sin x = \sin(n)x - \sin(n-2)x$$
$$2\cos nx \sin x = \sin(n+1)x - \sin(n-1)x$$

Adding the n equations above, we have

$$2\sin x \sum_{r=1}^{n} \cos rx = \sin(n+1)x + \sin nx - \sin x$$

$$2\sin x \sum_{r=1}^{n} \cos rx = 2\sin\left(\frac{2n+1}{2}\right)x \cos\tfrac{1}{2}x - \sin x$$

$$2\left(2\sin\tfrac{1}{2}x \cos\tfrac{1}{2}x\right)\sum_{r=1}^{n} \cos rx = 2\sin\left(n+\tfrac{1}{2}\right)x \cos\tfrac{1}{2}x - 2\sin\tfrac{1}{2}x \cos\tfrac{1}{2}x$$

$$2\sin\tfrac{1}{2}x \sum_{r=1}^{n} \cos rx = \sin\left(n+\tfrac{1}{2}\right)x - \sin\tfrac{1}{2}x$$

$$\sum_{r=1}^{n} \cos rx = \frac{\sin\left(n+\tfrac{1}{2}\right)x - \sin\tfrac{1}{2}x}{2\sin\tfrac{1}{2}x} \quad \text{(Shown)}$$

$$\cos^2\left(\frac{x}{2}\right)+\cos^2(x)+\cos^2\left(\frac{3x}{2}\right)+\ldots+\cos^2\left(\frac{11x}{2}\right)$$

$$=\frac{1+\cos x}{2}+\frac{1+\cos 2x}{2}+\frac{1+\cos 3x}{2}+\ldots+\frac{1+\cos 11x}{2}$$

$$=\frac{1}{2}\left(11+\sum_{r=1}^{11}\cos rx\right)$$

$$=\frac{1}{2}\left(11+\frac{\sin\left(11+\frac{1}{2}\right)x-\sin\frac{1}{2}x}{2\sin\frac{1}{2}x}\right)$$

$$=\frac{1}{2}\left(11+\frac{\sin\frac{23}{2}x}{2\sin\frac{1}{2}x}-\frac{1}{2}\right)$$

$$=\frac{1}{2}\left(\frac{\sin\frac{23}{2}x}{2\sin\frac{1}{2}x}+\frac{21}{2}\right)=\frac{1}{4}\left(\frac{\sin\frac{23}{2}x}{\sin\frac{1}{2}x}+21\right)$$

Marker's comments

Most students were able to make use of the given result and apply the method of differences to solve for $\sum_{r=1}^{n}\cos rx=\sum_{r=1}^{n}\frac{\sin(r+1)x-\sin(r-1)x}{2\sin x}=\frac{\sin(n+1)x+\sin nx-\sin x}{2\sin x}$. Thereafter, many students fail to apply the appropriate factor formula and double-angle formula to obtain the desired answer.

The second part of the question involves the use of double-angle formula to convert $\cos^2\left(\frac{r}{2}\right)$ into $\frac{\cos(rx)+1}{2}$, but many students chose to replace the index r by $\frac{r}{2}$, which would not allow them to achieve anything. Some students lost credit by failing to express their answer in the form as stated in the question.

Q3

(i)

$$\sum_{r=1}^{n} a_r = \sum_{r=1}^{n}(T_r - T_{r-1})$$
$$= T_1 - T_0$$
$$+ T_2 - T_1$$
$$+ T_3 - T_2$$
$$\vdots$$
$$+ T_{n-1} - T_{n-2}$$
$$+ T_n - T_{n-1}$$
$$= T_n - T_0$$
$$= T_n$$

ii)

Let $T_r = r^2\pi^{-r}$

Note $T_0 = 0$

$$T_r - T_{r-1} = r^2\pi^{-r} - (r-1)^2\pi^{-r+1}$$
$$= \pi^{-r}\left[r^2 - (r^2 - 2r + 1)\pi\right]$$
$$= \pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$$
$$= a_r$$

$\therefore$ From (i),

$$\sum_{r=1}^{n}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right] = \sum_{r=1}^{n} a_r$$
$$= T_n = n^2\pi^{-n}$$

iii)

$$\sum_{r=4}^{20}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$$
$$= \sum_{r=1}^{20}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right] - \sum_{r=1}^{3}\pi^{-r}\left[(1-\pi)r^2 + 2\pi r - \pi\right]$$
$$= 400\pi^{-20} - 9\pi^{-3}$$