A Level H2 Math

Sigma Notation Test 5

Q1

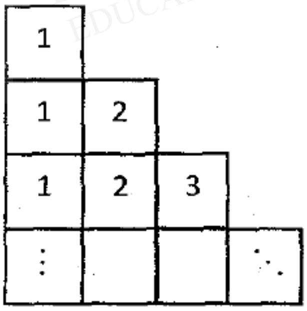
It is given that $\sum_{r=1}^{n} \frac{r^2}{3^r} = \frac{3}{2} - \frac{n^2+3n+3}{2(3^n)}$.

(i) Find $\sum_{r=1}^{\infty} \frac{r^2+(-1)^r}{3^r}$. [3]

(ii) Show that $\sum_{r=4}^{n} \frac{(r-2)^2}{3^{r-2}} = \frac{p}{q} - \frac{an^2-an+a}{2(3^{n-2})}$, where a, p and q are integers to be determined. [5]

Q2

A board is such that the n^{th} row from the top has n tiles, and each row is labelled from left to right in ascending order such that the i^{th} tile is labelled i, where n and i are positive integers.



Given that $\sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$, by finding the sum of the numbers in the r^{th} row, show that the sum of all the numbers in n rows of tiles is $\frac{1}{6}(n)(n+1)(n+2)$. [4]

Q3

(i) By considering $f(r)-f(r+1)$, where $f(r)=\frac{\sqrt{r}}{2\sqrt{r}+1}$, find

$$\sum_{r=1}^{n}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$$

in terms of n. [3]

(ii) Hence, find $\sum_{r=1}^{\infty}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$. [2]

(iii) Find the smallest integer n such that

$$\sum_{r=1}^{n}\frac{\sqrt{r+1}-\sqrt{r+2}}{\left(2\sqrt{r+1}+1\right)\left(2\sqrt{r+2}+1\right)}<-0.1.$$ [3]

Answers

Sigma Notation Test 5

Q1

i

$$\sum_{r=1}^{\infty}\frac{r^2+(-1)^r}{3^r}=\sum_{r=1}^{\infty}\frac{r^2}{3^r}+\sum_{r=1}^{\infty}\left(-\frac{1}{3}\right)^r$$

$$=\frac{3}{2}+\frac{\left(-\frac{1}{3}\right)}{1-\left(-\frac{1}{3}\right)}$$

$$=\frac{5}{4}$$

ii

$$\sum_{r=4}^{n}\frac{(r-2)^2}{3^{r-2}}=\sum_{r+2=4}^{r+2=n}\frac{r^2}{3^r}\quad(\text{replace } r \text{ with } r+2)$$

$$=\sum_{r=2}^{n-2}\frac{r^2}{3^r}$$

$$=\sum_{r=1}^{n-2}\frac{r^2}{3^r}-\frac{(1)^2}{3^1}$$

$$=\frac{3}{2}-\frac{(n-2)^2+3(n-2)+3}{2\left(3^{n-2}\right)}-\frac{1}{3}$$

$$=\frac{7}{6}-\frac{n^2-4n+4+3n-6+3}{2\left(3^{n-2}\right)}$$

$$=\frac{7}{6}-\frac{n^2-n+1}{2\left(3^{n-2}\right)}$$

$$\therefore p=7,\quad q=6,\quad a=1$$

Q2

Sum of numbers in kth row = $\sum_{r=1}^{k} r = \frac{1}{2}k(k+1)$

$$\text{Required sum} = \sum_{k=1}^{n} \frac{k(k+1)}{2}$$

$$= \frac{1}{2}\sum_{k=1}^{n}(k^2+k)$$

$$= \frac{1}{12}n(n+1)(2n+1) + \frac{1}{4}n(n+1)$$

$$= \frac{1}{12}n(n+1)(2n+1+3)$$

$$= \frac{1}{6}n(n+1)(n+2)$$

Q3

(i)

$$f(r)-f(r+1)=\frac{\sqrt{r}}{2\sqrt{r}+1}-\frac{\sqrt{r+1}}{2\sqrt{r+1}+1}$$
$$=\frac{\left(2\sqrt{r}\sqrt{r+1}+\sqrt{r}\right)-\left(2\sqrt{r+1}\sqrt{r}+\sqrt{r+1}\right)}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$$
$$=\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$$

$$\sum_{r=1}^{n}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}=\sum_{r=1}^{n}\left[f(r)-f(r+1)\right]$$
$$=f(1)-f(2)$$
$$+f(2)-f(3)$$
$$+\ldots$$
$$+f(n)-f(n+1)$$
$$=f(1)-f(n+1)$$
$$=\frac{1}{3}-\frac{\sqrt{n+1}}{2\sqrt{n+1}+1}$$

(ii)

$$\sum_{r=1}^{\infty}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$$
$$=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{\sqrt{n+1}}{2\sqrt{n+1}+1}\right)$$
$$=\lim_{n\to\infty}\left(\frac{1}{3}-\frac{1}{2+\frac{1}{\sqrt{n+1}}}\right)$$
$$=-\frac{1}{6}$$

Most students have the misconception that a fraction must go to 0 when the denominator goes to infinity. However, note that the numerator goes to infinity as well, thus the expression is indeterminate until further manipulation is done to get a clearer picture.

(iii)

$$\sum_{r=1}^{n}\frac{\sqrt{r+1}-\sqrt{r+2}}{\left(2\sqrt{r+1}+1\right)\left(2\sqrt{r+2}+1\right)}=\sum_{r=2}^{n+1}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}$$
$$=\sum_{r=1}^{n+1}\frac{\sqrt{r}-\sqrt{r+1}}{\left(2\sqrt{r}+1\right)\left(2\sqrt{r+1}+1\right)}-\frac{1-\sqrt{2}}{3(2\sqrt{2}+1)}$$
$$=\frac{1}{3}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}-\frac{1-\sqrt{2}}{3(2\sqrt{2}+1)}$$
$$=\frac{\sqrt{2}}{2\sqrt{2}+1}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}$$

One may alternatively input the expression as a summation into the GC, but the calculation of the sum for each n takes much longer.

Need $\frac{\sqrt{2}}{2\sqrt{2}+1}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}<-0.1$

n	$\frac{\sqrt{2}}{2\sqrt{2}+1}-\frac{\sqrt{n+2}}{2\sqrt{n+2}+1}$
56	-0.099797
57	-0.100043

Some students were not careful with the number of decimal places, thus unable to compare the value with -0.1.

Using GC, least $n=57$