A Level H2 Math

Sigma Notation Test 4

Q1

A sequence u_1, u_2, u_3,... is such that

$$u_n = \frac{1}{2n^2(n-1)^2} \text{ and } u_{n+1} = u_n - \frac{2}{n(n-1)^2(n+1)^2}, \text{ for all } n \geq 2 .$$

(i) Find $\sum_{n=2}^{N} \frac{2}{n(n-1)^2(n+1)^2}$. [3]

(ii) Explain why $\sum_{n=2}^{\infty} \frac{2}{n(n-1)^2(n+1)^2}$ is a convergent series, and state the value of the sum to infinity. [2]

(iii) Using your answer in part **(i)**, find $\sum_{n=1}^{N} \frac{2N}{(n+1)n^2(n+2)^2}$. [2]

Q2

(a) **(i)** Show that $\frac{1}{r-1}-\frac{2}{r}+\frac{1}{r+1}=\frac{2}{r(r-1)(r+1)}$. [1]

(ii) Hence find $\sum_{r=3}^{n}\frac{4}{r(r-1)(r+1)}$.

(There is no need to express your answer as a single algebraic fraction). [4]

(b) Amy and her brother Ben are saving money together for their family trip. In the first week of 2017, Amy saves $25 and Ben saves $2. In each subsequent week, Amy saves $4 more than the amount she saved in the previous week, and Ben saves 22% more than the amount he saved in the previous week.

(i) Which is the first week in which Ben saves more than Amy in that week? [2]

(ii) They need a combined total of $2400 for the trip. How many complete weeks do Amy and Ben need to save before they can achieve their targeted amount? [2]

Q3

The function f is defined by $\mathrm{f}: x \mapsto \frac{1}{x^2-1}, \quad x \in \mathbb{R}, \ x>1$.

(i) Show that $\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}=\frac{An+B}{n^3-n}$, where A and B are constants to be found. [3]

(ii) Hence find $\sum_{r=2}^{n}\frac{2r+6}{r^3-r}$. [4]

(iii) Use your answer to part **(ii)** to find $\sum_{r=2}^{n}\frac{2r+10}{(r+1)(r+2)(r+3)}$. [1]

Answers

Sigma Notation Test 4

Q1

(i)

$$\sum_{n=2}^{N} \frac{2}{n(n-1)^2(n+1)^2}$$

$$= \sum_{n=2}^{N} [u_n - u_{n+1}]$$

$$= \left[\begin{array}{l} (u_2 - u_3) \\ +(u_3 - u_4) \\ +(u_4 - u_5) \\ \cdots\cdots \\ \cdots\cdots \\ +(u_{N-1} - u_N) \\ +(u_N - u_{N+1}) \end{array} \right]$$

$$= u_2 - u_{N+1}$$

$$= \frac{1}{2(2^2)(2-1)^2} - \frac{1}{2(N+1)^2((N-1)+1)^2}$$

$$= \frac{1}{8} - \frac{1}{2N^2(N+1)^2}$$

(ii)

As $N \to \infty$, $\frac{1}{2N^2(N+1)^2} \to 0$

$\sum_{n=2}^{\infty} \frac{2}{n(n-1)^2(n+1)^2} \to \frac{1}{8}$ which is a constant, hence it is a convergent series.

$$\sum_{n=2}^{\infty} \frac{2}{n(n-1)^2(n+1)^2} = \frac{1}{8} - 0$$

$$= \frac{1}{8}$$

(iii)

Method 1

$$\sum_{n=1}^{N}\frac{2N}{(n+1)n^2(n+2)^2}=N\sum_{n=1}^{N}\frac{2}{(n+1)n^2(n+2)^2}$$

$$=N\sum_{n=2}^{N+1}\frac{2}{(n)(n-1)^2(n+1)^2}$$

$$=N\left[\frac{1}{8}-\frac{1}{2(N+1)^2(N+2)^2}\right]$$

$$=\frac{N}{8}\left[1-\frac{4}{(N+1)^2(N+2)^2}\right]$$

Method 2 By listing the terms

$$\sum_{n=2}^{N}\frac{2}{n(n-1)^2(n+1)^2}$$

$$=\frac{2}{2(1)^2(3)^2}+\frac{2}{3(2)^2(4)^2}+\cdots+\frac{2}{N(N-1)^2(N+1)^2}$$

$$\sum_{n=1}^{N}\frac{2N}{(n+1)n^2(n+2)^2}$$

$$=N\left[\frac{2}{2(1)^2(3)^2}+\frac{2}{3(2)^2(4)^2}+\cdots+\frac{2}{(N+1)(N)^2(N+2)^2}\right]$$

$$=N\sum_{n=2}^{N+1}\frac{2}{n(n-1)^2(n+1)^2}$$

$$=N\left[\frac{1}{8}-\frac{1}{2(N+1)^2(N+2)^2}\right]$$

$$=\frac{N}{8}\left[1-\frac{4}{(N+1)^2(N+2)^2}\right]$$

Q2

(a)(i)

$$\frac{1}{r-1}-\frac{2}{r}+\frac{1}{r+1}=\frac{r(r+1)-2(r-1)(r+1)+r(r-1)}{r(r-1)(r+1)}$$

$$=\frac{r^2+r-2(r^2-1)+r^2-r}{r(r-1)(r+1)}$$

$$=\frac{2}{r(r-1)(r+1)}$$

(a)(ii)

$$\sum_{r=3}^{n}\frac{4}{r(r-1)(r+1)}=2\sum_{r=3}^{n}\frac{2}{r(r-1)(r+1)}$$

$$=2\sum_{r=3}^{n}\left(\frac{1}{r-1}-\frac{2}{r}+\frac{1}{r+1}\right)$$

$$=2\Big[\frac{1}{2}-\frac{2}{3}+\frac{1}{4}$$
$$+\frac{1}{3}-\frac{2}{4}+\frac{1}{5}$$
$$+\frac{1}{4}-\frac{2}{5}+\frac{1}{6}$$
$$+\frac{1}{5}-\frac{2}{6}+\frac{1}{7}$$
$$+\ldots$$
$$+\frac{1}{n-3}-\frac{2}{n-2}+\frac{1}{n-1}$$
$$+\frac{1}{n-2}-\frac{2}{n-1}+\frac{1}{n}$$
$$+\frac{1}{n-1}-\frac{2}{n}+\frac{1}{n+1}\Big]$$

$$=2\left(\frac{1}{6}-\frac{1}{n}+\frac{1}{n+1}\right)=\frac{1}{3}-\frac{2}{n}+\frac{2}{n+1}$$

(b)(i)

Amount Amy saves in nth week $=25+(n-1)(4)=21+4n$

Amount Ben saves in nth week $=ar^{n-1}=2(1.22)^{n-1}$

When Ben saves more than Amy,

$$2(1.22)^{n-1}>21+4n$$

From GC,

in the 20th week, Amy saves \$101, Ben saves \$87.47

in the 21st week, Amy saves \$105, Ben saves \$106.72

Hence, Ben first saves more than Amy in the 21st week.

Or: $2(1.22)^{n-1} > 21+4n \quad \Rightarrow 2(1.22)^{n-1} - 21 - 4n > 0$

When $n = 20$, $2(1.22)^{n-1} - 21 - 4n = -13.5 < 0$

When $n = 21$, $2(1.22)^{n-1} - 21 - 4n = 1.72 > 0$

(b)(ii) Total amount in Amy's account after nth week,

$$= \frac{n}{2}\left(2a + (n-1)d\right) = \frac{n}{2}\left(50 + (n-1)(4)\right) = \frac{n}{2}(46 + 4n)$$

Total amount in Ben's account after nth week,

$$= \frac{a\left(r^n - 1\right)}{r-1} = \frac{2\left(1.22^n - 1\right)}{1.22 - 1}$$

For their total saving to exceed \$2400,

$$\frac{n}{2}(46 + 4n) + \frac{2\left(1.22^n - 1\right)}{1.22 - 1} > 2400$$

From GC,

in the 22nd week, total savings= \$2186.89 < \$2400

in the 23rd week, total savings= \$2458.72 > \$2400

$\therefore$ least $n = 23$.

Hence, their total saving will exceed \$2400 after 23 complete weeks.

Q3

(i)

$$\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}$$
$$=\frac{2(n)(n+1)-3(n-1)(n+1)+(n-1)(n)}{(n-1)(n)(n+1)}$$
$$=\frac{(2n^2+2n)-(3n^2-3)+(n^2-n)}{n^3-n}$$
$$=\frac{n+3}{n^3-n}$$

(ii)

$$\sum_{r=2}^{n}\frac{2r+6}{r^3-r}$$
$$=2\sum_{r=2}^{n}\frac{r+3}{r^3-r}$$
$$=2\sum_{r=2}^{n}\left(\frac{2}{r-1}-\frac{3}{r}+\frac{1}{r+1}\right)$$
$$=2\left[\begin{array}{l}\frac{2}{1}-\frac{3}{2}+\frac{1}{3}\\+\frac{2}{2}-\frac{3}{3}+\frac{1}{4}\\+\frac{2}{3}-\frac{3}{4}+\frac{1}{5}\\+\ldots\\+\frac{2}{n-2}-\frac{3}{n-1}+\frac{1}{n}\\+\frac{2}{n-1}-\frac{3}{n}+\frac{1}{n+1}\end{array}\right]$$

$$= 2\left(\frac{2}{1} - \frac{3}{2} + \frac{2}{2} + \frac{1}{n} - \frac{3}{n} + \frac{1}{n+1}\right)$$

$$= 2\left(\frac{3}{2} - \frac{2}{n} + \frac{1}{n+1}\right)$$

$$= 3 - \frac{4}{n} + \frac{2}{n+1}$$

(iii)

$$\sum_{r=2}^{n} \frac{2r+10}{(r+1)(r+2)(r+3)}$$

Let $r+2 = p \Rightarrow r = p-2$

$$= \sum_{p-2=2}^{p-2=n} \frac{2p+6}{(p-1)(p)(p+1)}$$

$$= \sum_{p=4}^{n+2} \frac{2p+6}{p^3 - p}$$

$$= \sum_{p=2}^{n+2} \frac{2p+6}{p^3 - p} - \sum_{p=2}^{3} \frac{2p+6}{p^3 - p}$$

$$= \left(3 - \frac{4}{n+2} + \frac{2}{n+3}\right) - \left(3 - \frac{4}{3} + \frac{2}{4}\right)$$

$$= \frac{5}{6} - \frac{4}{n+2} + \frac{2}{n+3}$$