A Level H2 Math

Sigma Notation Test 3

Q1

(i) Using the formula for $\sin P - \sin Q$, show that

$$\sin\left[(2r+1)\theta\right] - \sin\left[(2r-1)\theta\right] \equiv 2\cos(2r\theta)\sin\theta.$$ [1]

(ii) Given that $\sin\theta \neq 0$, using the method of differences, show that

$$\sum_{r=1}^{n}\cos(2r\theta) = \frac{\sin\left[(2n+1)\theta\right] - \sin\theta}{2\sin\theta}.$$ [2]

(iii) Hence find $\sum_{r=1}^{n}\cos^2\left(\frac{r\pi}{5}\right)$ in terms of n.

Explain why the infinite series

$$\cos^2\left(\frac{\pi}{5}\right) + \cos^2\left(\frac{2\pi}{5}\right) + \cos^2\left(\frac{3\pi}{5}\right) + \ldots$$

is divergent. [3]

Q2

(a) Given that $\sum_{n=1}^{N}\frac{1}{4n^2-1} = \frac{1}{2} - \frac{1}{2(2N+1)}$, find $\sum_{n=1}^{2N}\frac{1}{4(n+1)^2-1}$.

Deduce that $\sum_{n=1}^{2N}\frac{1}{(2n+3)^2}$ is less than $\frac{1}{6}$. [5]

(b) The sum to n terms of a series is given by $S_n = n\ln 2 - \frac{n^2-1}{e}$.

Find an expression for the n^{th} term of the series, in terms of n.
Show that the terms of the series follow an arithmetic progression. [4]

Q3

(i) Prove that $\dfrac{\sin(A-B)}{\cos A\cos B}=\tan A-\tan B$. [1]

(ii) Hence, by considering a suitable expression of A and B, find

$$\sum_{r=1}^{N}\frac{\sin x}{\cos\left[(r+1)x\right]\cos(rx)}.$$ [3]

(iii) Using your answer to part **(ii)**, find $\displaystyle\sum_{r=1}^{N}\left(\frac{\sqrt{3}}{2\cos\frac{r\pi}{3}\cos\frac{(r+1)\pi}{3}}\right)$, leaving your answer in terms of N. [2]

Answers

Sigma Notation Test 3

Q1

(i)

$$\sin\left[(2r+1)\theta\right]-\sin\left[(2r-1)\theta\right]$$
$$\equiv 2\cos\frac{(2r+1)\theta+(2r-1)\theta}{2}\sin\frac{(2r+1)\theta-(2r-1)\theta}{2}$$
$$\equiv 2\cos(2r\theta)\sin\theta \quad \text{[Shown]}$$

(ii) From (i), $\sin\left[(2r+1)\theta\right]-\sin\left[(2r-1)\theta\right]\equiv 2\cos(2r\theta)\sin\theta$

$$\Rightarrow \cos(2r\theta)=\frac{\sin\left[(2r+1)\theta\right]-\sin\left[(2r-1)\theta\right]}{2\sin\theta}$$

$$\therefore \sum_{r=1}^{n}\cos(2r\theta) = \sum_{r=1}^{n}\frac{\sin\left[(2r+1)\theta\right]-\sin\left[(2r-1)\theta\right]}{2\sin\theta}$$

$$=\frac{1}{2\sin\theta}\left[\begin{array}{l} \sin 3\theta-\sin\theta \\ +\ \sin 5\theta-\sin 3\theta \\ +\ \sin 7\theta-\sin 5\theta \\ +\ \cdots \\ +\ \sin(2n-1)\theta-\sin(2n-3)\theta \\ +\ \sin(2n+1)\theta-\sin(2n-1)\theta \end{array}\right]$$

$$=\frac{\sin\left[(2n+1)\theta\right]-\sin\theta}{2\sin\theta} \quad \text{[Shown]}$$

(iii)
$$\sum_{r=1}^{n}\cos^2\left(\frac{r\pi}{5}\right) = \sum_{r=1}^{n}\frac{\cos\left(\frac{2r\pi}{5}\right)+1}{2}$$

$$= \frac{1}{2}\sum_{r=1}^{n}\cos\left(\frac{2r\pi}{5}\right)+\sum_{r=1}^{n}\frac{1}{2} \qquad \left(\text{Let } \theta=\frac{\pi}{5}\right)$$

$$= \frac{1}{2}\left[\frac{\sin\frac{(2n+1)\pi}{5}-\sin\frac{\pi}{5}}{2\sin\frac{\pi}{5}}\right]+\frac{1}{2}n$$

$$= \frac{\sin\frac{(2n+1)\pi}{5}}{4\sin\frac{\pi}{5}}-\frac{1}{4}+\frac{1}{2}n$$

As $n\to\infty$, $-\frac{1}{4}+\frac{1}{2}n\to\infty$ and $\left|\sin\frac{(2n+1)\pi}{5}\right|\le 1$,

$$\therefore \sum_{r=1}^{n}\cos^2\left(\frac{r\pi}{5}\right)\to\infty.$$

$\therefore$ the series $\cos^2\left(\frac{\pi}{5}\right)+\cos^2\left(\frac{2\pi}{5}\right)+\cos^2\left(\frac{3\pi}{5}\right)+\ldots$ is divergent.

Q2

(a)

$$\sum_{n=1}^{2N}\frac{1}{4(n+1)^2-1}=\sum_{n=2}^{2N+1}\frac{1}{4n^2-1}$$

$$=\sum_{n=1}^{2N+1}\frac{1}{4n^2-1}-\frac{1}{3}$$

$$=\frac{1}{2}-\frac{1}{2\left[2(2N+1)+1\right]}-\frac{1}{3}$$

$$=\frac{1}{6}-\frac{1}{2(4N+3)}$$

$$\frac{1}{(2n+3)^2}=\frac{1}{4n^2+12n+9} \quad \text{and} \quad \frac{1}{4(n+1)^2-1}=\frac{1}{4n^2+8n+3}$$

$$\therefore \quad \frac{1}{(2n+3)^2}<\frac{1}{4(n+1)^2-1}$$

Alternative:

$$\frac{1}{(2n+3)^2}<\frac{1}{(2n+1)(2n+3)}=\frac{1}{4(n+1)^2-1}$$

Hence

$$\sum_{n=1}^{2N}\frac{1}{(2n+3)(2n+3)}<\sum_{n=1}^{2N}\frac{1}{4(n+1)^2-1}$$

$$\sum_{n=1}^{2N}\frac{1}{(2n+3)(2n+3)}<\frac{1}{6}-\frac{1}{2(4N+3)}$$

$$<\frac{1}{6}$$

[since N>0 & $\frac{1}{2(4N+3)}>0$]

b

$$T_n = S_n - S_{n-1} = n\ln 2 - \frac{n^2-1}{e} - \left[(n-1)\ln 2 - \frac{(n-1)^2-1}{e}\right]$$

$$= \left[n-(n-1)\right]\ln 2 - \frac{1}{e}\left[\left(n^2-1\right)-(n-1)^2+1\right]$$

$$= \ln 2 - \frac{1}{e}\left[n^2-1-n^2+2n-1+1\right]$$

$$= \ln 2 - \frac{1}{e}(2n-1)$$

$$T_n - T_{n-1} = \ln 2 - \frac{1}{e}(2n-1) - \left[\ln 2 - \frac{1}{e}\left(2(n-1)-1\right)\right]$$

$$= -\frac{2}{e}$$

Since $-\frac{2}{e}$ is a constant, the terms follow an AP.

Q3

(i)

$$\frac{\sin(A-B)}{\cos A\cos B}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B}=\frac{\sin A}{\cos A}-\frac{\sin B}{\cos B}=\tan A-\tan B$$

(ii)

$$\sum_{r=1}^{N}\frac{\sin x}{\cos(r+1)x\cos rx}=\frac{\sin(2x-x)}{\cos 2x\cos x}+\frac{\sin(3x-2x)}{\cos 3x\cos 2x}+\frac{\sin(4x-3x)}{\cos 4x\cos 3x}+\ldots+\frac{\sin((N+1)x-Nx)}{\cos(N+1)x\cos Nx}$$

$$\begin{aligned}
&= (\tan 2x-\tan x)\\
&+ (\tan 3x-\tan 2x)\\
&+ (\tan 4x-\tan 3x)\\
&\quad\vdots\\
&+ (\tan(N-1)x-\tan(N-2)x\\
&+ (\tan Nx-\tan(N-1)x\\
&+ (\tan(N+1)x-\tan Nx\\
&= \tan(N+1)x-\tan x
\end{aligned}$$

(iii)

When $x=\frac{\pi}{3}$, $\sum_{r=1}^{N}\frac{\sin x}{\cos(r+1)x\cos rx}=\sum_{r=1}^{N}\left(\frac{\sqrt{3}}{2\cos\frac{r\pi}{3}\cos\frac{(r+1)\pi}{3}}\right)$

Thus, required sum = $\tan\left[(N+1)\left(\frac{\pi}{3}\right)\right]-\tan\left(\frac{\pi}{3}\right)=\tan\left[\frac{(N+1)\pi}{3}\right]-\sqrt{3}$