A Level H2 Math

Sigma Notation Test 2

Q1

(a) The sum, S_n, of the first n terms of a sequence $u_1, u_2, u_3, \ldots$ is given by $S_n = 3 + 7^{-2n}\left(n^2\right)$.

(i) Write down the value of $\sum_{r=1}^{\infty} u_r$. [1]

(ii) Find a formula for u_n for $n \geq 2$ and leave it in the form $7^{-2n}\,\mathrm{g}(n)$, where $\mathrm{g}(n)$ is an expression in terms of n. [2]

(b) Show that $\sum_{r=1}^{n}\left(\int_0^r \mathrm{e}^x - \mathrm{e}^{x-1}\,\mathrm{d}x\right) = \mathrm{e}^n + n\mathrm{e}^{-1} - (n+1)$.

Deduce the exact value of $\sum_{r=10}^{20}\left(\int_0^r \mathrm{e}^{x+2} - \mathrm{e}^{x+1}\,\mathrm{d}x\right)$. [5]

Q2

Given that $\sum_{k=1}^{n} k!(k^2+1) = (n+1)!\,n$, find $\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)$. [3]

Q3

A geometric sequence $T_1, T_2, T_3, \ldots$ has a common ratio of e. Another sequence $U_1, U_2, U_3, \ldots$ is such that $U_1 = 1$ and

$$U_r = \ln T_r - 3 \quad \text{for all } r \geq 1.$$

(i) Prove that the sequence $U_1, U_2, U_3, \ldots$ is arithmetic. [2]

A third sequence $W_1, W_2, W_3, \ldots$ is such that $W_1 = \frac{1}{2}$ and $W_{r+1} = W_r + U_r$ for all $r \geq 1$.

(ii) By considering $\sum_{r=1}^{n-1}(W_{r+1} - W_r)$, show that $W_n = \frac{1}{2}(n^2 - n + 1)$. [3]

Answers

Sigma Notation Test 2

Q1

(a) (i)

By GC, sum to infinity is 3.

(a) (ii)

$$u_n = S_n - S_{n-1}$$
$$= 3 + 7^{-2n}\left(n^2\right) - \left[3 + 7^{-2(n-1)}\left(n-1\right)^2\right]$$
$$= 3 - 3 + 7^{-2n}\left(n^2\right) - 7^{-2n+2}\left(n^2 - 2n + 1\right)$$
$$= 7^{-2n}\left(n^2 - 49n^2 + 98n - 49\right)$$
$$= 7^{-2n}\left(-48n^2 + 98n - 49\right)$$
$$= 7^{-2n}\left(8n-7\right)\left(7-6n\right)$$

where $g(n) = -48n^2 + 98n - 49$

(b)

$$\sum_{r=1}^{n}\left(\int_0^r e^x - e^{x-1}\,dx\right)$$
$$= \sum_{r=1}^{n}\left[e^x - e^{x-1}\right]_0^r$$
$$= \sum_{r=1}^{n}\left(e^r - e^{r-1} - e^0 + e^{-1}\right)$$
$$= e^1 - e^0 - e^0 + e^{-1}$$
$$+e^2 - e^1 - e^0 + e^{-1}$$
$$+e^3 - e^2 - e^0 + e^{-1}$$
$$\ldots$$
$$+e^n - e^{n-1} - e^0 + e^{-1}$$
$$= e^n - 1 - n(1) + ne^{-1}$$
$$= e^n + ne^{-1} - (n+1)$$

$$\sum_{r=10}^{20}\left(\int_0^r \mathrm{e}^{x+2}-\mathrm{e}^{x+1}\ \mathrm{d}x\right)$$

$$=\mathrm{e}^2\sum_{r=1}^{20}\left(\int_0^r \mathrm{e}^{x}-\mathrm{e}^{x-1}\ \mathrm{d}x\right)-\mathrm{e}^2\sum_{r=1}^{9}\left(\int_0^r \mathrm{e}^{x}-\mathrm{e}^{x-1}\ \mathrm{d}x\right)$$

$$=\mathrm{e}^2\left[\mathrm{e}^{20}+20\mathrm{e}^{-1}-(20+1)-\left(\mathrm{e}^{9}+9\mathrm{e}^{-1}-10\right)\right]$$

$$=\mathrm{e}^{22}-\mathrm{e}^{11}-11\mathrm{e}^2+11\mathrm{e}$$

Q2

Method 1

Consider replace k by $(k-1)$:

$$\sum_{k=1}^{n-1}(k+1)!(k^2+2k+2)=\sum_{k-1=1}^{k-1=n-1}(k-1+1)!((k-1)^2+2(k-1)+2)$$

$$=\sum_{k=2}^{n}k!(k^2+1)$$

$$=\sum_{k=1}^{n}k!(k^2+1)-1!(1^2+1)$$

$$=(n+1)!n-2$$

Method 2

$$\sum_{k=1}^{n}k!\left(k^2+1\right)=\sum_{k=0}^{n-1}(k+1)!\left((k+1)^2+1\right)$$

$$=\sum_{k=0}^{n-1}(k+1)!\left(k^2+2k+2\right)$$

$$=(n+1)!n$$

$$\sum_{k=1}^{n-1}(k+1)!\left(k^2+2k+2\right)=\sum_{k=0}^{n-1}(k+1)!\left(k^2+2k+2\right)$$

$$+(0+1)!\left(0^2+2(0)+2\right)$$

$$=(n+1)!n+2$$

Q3

(i) To prove AP, consider

$U_{r+1} - U_r$

$= (\ln T_{r+1} - 3) - (\ln T_r - 3)$

$= \ln\left(\frac{T_{r+1}}{T_r}\right)$

$= \ln e$

$= 1$

Since difference is a **constant,** the sequence is arithmetic. (Proven)

$$\sum_{r=1}^{n-1}(W_{r+1} - W_r) = \sum_{r=1}^{n-1} U_r$$

$$\text{LHS} = \sum_{r=1}^{n-1}(W_{r+1} - W_r)$$

$= W_2 - W_1$

$+ W_3 - W_2$

$+ W_4 - W_3$

$\vdots$

$+ W_n - W_{n-1}$

$= W_n - W_1$

$= W_n - \frac{1}{2}$

$$\text{RHS} = \sum_{r=1}^{n-1} U_r$$

$= U_1 + U_2 + \ldots + U_{n-1}$

$= \frac{n-1}{2}(2(1) + (n-2)1)$

$= \frac{n(n-1)}{2}$

Thus, $W_n - \frac{1}{2} = \frac{n(n-1)}{2}$

$\therefore W_n = \frac{1}{2}\left(n^2 - n + 1\right)$ (shown)