A Level H2 Math

Sigma Notation Test 1

Q1

(i) Express $\frac{1}{r^2-1}$ in partial fractions, and deduce that

$$\frac{1}{r(r^2-1)}=\frac{1}{2}\left[\frac{1}{r(r-1)}-\frac{1}{r(r+1)}\right].$$ [2]

(ii) Hence, find the sum, S_n, of the first n terms of the series

$$\frac{1}{2\cdot 3}+\frac{1}{3\cdot 8}+\frac{1}{4\cdot 15}+\ldots .$$ [4]

(iii) Explain why the series converges, and write down the value of the sum to infinity. [2]

(iv) Find the smallest value of n for which S_n is smaller than the sum to infinity by less than 0.0025. [3]

Q2

(i) Express $\frac{4r+6}{(r+1)(r+2)(r+3)}$ as partial fractions. [1]

(ii) Hence find $\sum_{r=1}^{n}\frac{4r+6}{(r+1)(r+2)(r+3)}$ in terms of n. [3]

(iii) Use your answer in part **(ii)** to find the sum of the infinite series

$$\frac{3}{1\times2\times3}+\frac{5}{2\times3\times4}+\frac{7}{3\times4\times5}+\frac{9}{4\times5\times6}+\cdots.$$ [3]

Q3

(i) Express $\frac{r+1}{(r+2)!}$ in the form $\frac{A}{(r+1)!}+\frac{B}{(r+2)!}$, where A and B are integers to be found. [2]

(ii) Find $\sum_{r=1}^{n}\frac{r+1}{3(r+2)!}$. [3]

(iii) Hence, evaluate $\sum_{r=0}^{\infty}\frac{r+1}{3(r+2)!}$. [2]

Answers

Sigma Notation Test 1

Q1

(i)

$$\frac{1}{r^2-1}=\frac{1}{2(r-1)}-\frac{1}{2(r+1)}$$

$$\frac{1}{r(r^2-1)}=\frac{1}{r}\left[\frac{1}{2(r-1)}-\frac{1}{2(r+1)}\right]$$

$$=\frac{1}{2}\left[\frac{1}{r(r-1)}-\frac{1}{r(r+1)}\right]$$

(ii)

$$S_n=\frac{1}{2\times 3}+\frac{1}{3\times 8}+\frac{1}{4\times 15}+\ldots+(n\text{th term})$$

$$=\sum_{r=2}^{n+1}\frac{1}{r(r^2-1)}$$

$$=\frac{1}{2}\sum_{r=2}^{n+1}\left[\frac{1}{r(r-1)}-\frac{1}{r(r+1)}\right]$$

$$=\frac{1}{2}\left[\frac{1}{2\times 1}-\frac{1}{2\times 3}\right.$$
$$+\frac{1}{3\times 2}-\frac{1}{3\times 4}$$
$$+\frac{1}{4\times 3}-\frac{1}{4\times 5}$$
$$\bullet$$
$$\bullet$$
$$\bullet$$
$$+\frac{1}{(n-1)\times(n-2)}-\frac{1}{(n-1)\times n}$$
$$+\frac{1}{(n)\times(n-1)}-\frac{1}{n\times(n+1)}$$
$$\left.+\frac{1}{(n+1)\times n}-\frac{1}{(n+1)\times(n+2)}\right]$$

$$=\frac{1}{2}\left[\frac{1}{2}-\frac{1}{(n+1)(n+2)}\right]$$

$$=\frac{1}{4}-\frac{1}{2(n+1)(n+2)}$$

iii)	As $n \to \infty$, $\frac{1}{2(n+1)(n+2)} \to 0$. $\frac{1}{4} - \frac{1}{2(n+1)(n+2)} \to \frac{1}{4}$ Sum to infinity $= \frac{1}{4}$
iv)	$(0 <)\frac{1}{4} - S_n < 0.0025$ $\Rightarrow (0 <)\frac{1}{4} - \left[\frac{1}{4} - \frac{1}{2(n+1)(n+2)}\right] < 0.0025$ $\Rightarrow (0 <)\frac{1}{2(n+1)(n+2)} < 0.0025$ $\Rightarrow (n+1)(n+2) > 200$ Using G.C. $n < -15.651$ or $n > 12.651$ Since $n \in \mathbb{Z}^+$, Smallest value of $n = 13$

Q2

(i)

Let $\dfrac{4r+6}{(r+1)(r+2)(r+3)}=\dfrac{A}{r+1}+\dfrac{B}{r+2}+\dfrac{C}{r+3}$

Then by cover up rule, $A=1, B=2, C=-3$

Hence, $\dfrac{4r+6}{(r+1)(r+2)(r+3)}=\dfrac{1}{r+1}+\dfrac{2}{r+2}-\dfrac{3}{r+3}$

(ii)

$$\sum_{r=1}^{n}\frac{4r+6}{(r+1)(r+2)(r+3)}=\sum_{r=1}^{n}\frac{1}{r+1}+\frac{2}{r+2}-\frac{3}{r+3}$$

$$\begin{aligned}
&= \frac{1}{2} + \frac{2}{3} - \frac{3}{4} \\
&+ \frac{1}{3} + \frac{2}{4} - \frac{3}{5} \\
&+ \frac{1}{4} + \frac{2}{5} - \frac{3}{6} \\
&+ \frac{1}{5} + \ldots\ldots \quad \ldots\ldots \\
&+ \ldots\ldots \quad \ldots\ldots - \frac{3}{n} \\
&+ \frac{1}{n-1} + \frac{2}{n} - \frac{3}{n+2} \\
&+ \frac{1}{n} + \frac{2}{n+1} - \frac{3}{n+2} \\
&+ \frac{1}{n+1} + \frac{2}{n+2} - \frac{3}{n+3} \\
&= \frac{1}{2}+\frac{2}{3}+\frac{1}{3}-\frac{3}{n+2}+\frac{2}{n+2}-\frac{3}{n+3} \\
&= \frac{3}{2}-\left(\frac{1}{n+2}+\frac{3}{n+3}\right)
\end{aligned}$$

(iii)

$$\frac{3}{1\times2\times3}+\frac{5}{2\times3\times4}+\frac{7}{3\times4\times5}+\frac{9}{4\times5\times6}+\ldots$$

$$=\frac{3}{1\times2\times3}+\frac{1}{2}\left(\frac{10}{2\times3\times4}+\frac{14}{3\times4\times5}+\frac{18}{4\times5\times6}+\ldots\right)$$

$$=\frac{1}{2}+\frac{1}{2}\sum_{r=1}^{\infty}\frac{4r+6}{(r+1)(r+2)(r+3)}$$

$$=\frac{1}{2}+\frac{1}{2}\lim_{n\to\infty}\left(\frac{3}{2}-\left(\frac{1}{n+2}+\frac{3}{n+3}\right)\right)$$

$$=\frac{1}{2}+\frac{1}{2}\times\frac{3}{2}$$

$$=\frac{5}{4}$$

Q3

(i) **Method ①:**

$$\frac{r+1}{(r+2)!}=\frac{A}{(r+1)!}+\frac{B}{(r+2)!}$$

$$r+1=\frac{A(r+2)!}{(r+1)!}+\frac{B(r+2)!}{(r+2)!}$$

$$r+1=A(r+2)+B$$

When $r=-1$, $A+B=0$ — ①

When $r=0$, $2A+B=1$ — ②

Solving, $A=1$ and $B=-1$

$$\therefore \frac{1}{(r+2)r!}=\frac{1}{(r+1)!}-\frac{1}{(r+2)!}$$

Method ②:

$$\frac{r+1}{(r+2)!}=\frac{A}{(r+1)!}+\frac{B}{(r+2)!}$$

When $r=1$, $\frac{A}{2}+\frac{B}{6}=\frac{2}{6}$

$3A+B=2$ — ①

When $r=0$, $A+\frac{B}{2}=\frac{1}{2}$

$2A+B=1$ — ②

Solving, $A=1$ and $B=-1$

$$\therefore \frac{1}{(r+2)r!}=\frac{1}{(r+1)!}-\frac{1}{(r+2)!}$$

Most students were able to get the values of A and B correctly. There were a variety of methods used to get the correct answers.

(ii)

$$\sum_{r=1}^{n}\frac{r+1}{3(r+2)!}=\frac{1}{3}\sum_{r=1}^{n}\frac{r+1}{(r+2)!}$$

$$=\frac{1}{3}\sum_{r=1}^{n}\left[\frac{1}{(r+1)!}-\frac{1}{(r+2)!}\right]$$

$$=\frac{1}{3}\Bigg[\frac{1}{2!}-\frac{1}{3!}$$

$$+\frac{1}{3!}-\frac{1}{4!}$$

$$+\ \dots$$

$$+\frac{1}{n!}-\frac{1}{(n+1)!}$$

$$+\frac{1}{(n+1)!}-\frac{1}{(n+2)!}\Bigg]$$

$$=\frac{1}{3}\left[\frac{1}{2}-\frac{1}{(n+2)!}\right]$$

Most students were able to get this part correct.

(iii)

From (ii), $\sum_{r=1}^{n}\frac{r+1}{3(r+2)!}=\frac{1}{3}\left[\frac{1}{2}-\frac{1}{(n+2)!}\right]$

As $n\to\infty$, $\frac{1}{(n+2)!}\to 0$, thus $\sum_{r=1}^{n}\frac{r+1}{3(r+2)!}\to\frac{1}{6}$

$$\sum_{r=0}^{\infty}\frac{r+1}{3(r+2)!}=\frac{1}{3(2)(1)}+\sum_{r=1}^{\infty}\frac{r+1}{3(r+2)!}$$

$$=\frac{1}{6}+\frac{1}{6}$$

$$=\frac{1}{3}$$

Most students were able to get the sum to infinity correct but failed to realized that the starting value of r had change.