A Level H2 Math

Permutations, Combinations and Probability Test 3

Q1

Mandy has 10 beads, of which 5 are spherical and 5 are cubical, each of different colours. She wishes to decorate a card by forming a circle using 8 of the 10 beads. Find the number of ways Mandy can arrange the beads if

(i) there are no restrictions, [1]

(ii) 3 particular beads are used and not all are next to one another, [3]

(iii) spherical beads and cubic beads must alternate. [3]

Q2

A man wishes to buy a 4-digit number lottery. He plays by randomly choosing any number from 0000 to 9999. It is assumed that each number is equally likely to be chosen. Find the probability that a randomly chosen 4-digit number has

(i) four different digits, [1]

(ii) exactly one of the first three digits is the same as the last digit, and the last digit is even, [3]

(iii) four different digits with the first digit greater than 6, given that the 4-digit number has odd and even digits that alternate. [4]

Q3

(i) Find the number of 3-digit numbers that can be formed using the digits 1, 2 and 3 when

(a) no repetitions are allowed, [1]

(b) any repetitions are allowed, [1]

(c) each digit may be used at most twice. [2]

(ii) Find the number of 4-digit numbers that can be formed using the digits 1, 2 and 3 when each digit may be used at most twice. [5]

Answers
Permutations, Combinations and Probability Test 3

Q1

(i)

No. of ways $= {}^{10}C_8\,(8-1)! = 226800$

(ii)

Method 1: (method of complementation)

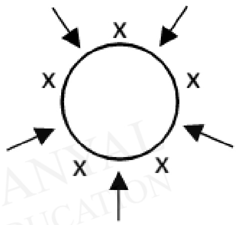
$$\text{No. of ways} = \underbrace{{}^{7}C_5\,(8-1)!}_{\substack{\text{No. of ways without restriction}\\ \text{[5 other beads, and together with 3}\\ \text{particular beads arranged in a circle]}}} - \underbrace{{}^{7}C_5\,3!(6-1)!}_{\substack{\text{No. of ways with 3 particular beads all together}\\ \text{[5 other beads, with 3 particular beads grouped}\\ \text{as 1 unit and 3! ways to arrange among themselves,}\\ \text{and all 6 units arranged in a circle]}}} = 90720$$

Method 2: (method of slotting)

Case 1: (all not next to one another)

No. of ways

$$= \underbrace{{}^{7}C_5\,(5-1)!}_{\substack{\text{5 other beads used}\\ \text{as seperators, and}\\ \text{arranged in a circle}}} \times \underbrace{{}^{5}C_3\,3!}_{\substack{\text{3 out of 5 slots}\\ \text{for 3 particular}\\ \text{beads, and 3! ways}\\ \text{to arrange among}\\ \text{themselves}}} = 30240$$



Case 2: (2 together, 1 not)

$$\text{No. of ways} = \underbrace{{}^{7}C_5\,(5-1)!}_{\substack{\text{5 other beads used}\\ \text{as seperators, and}\\ \text{arranged in a circle}}} \times \underbrace{{}^{3}C_2\,2!}_{\substack{\text{2 of 3 particular}\\ \text{beads together,}\\ \text{and 2! ways to}\\ \text{arrange among}\\ \text{themselves}}} \times \underbrace{{}^{5}C_2\,2!}_{\substack{\text{2 out of 5 slots}\\ \text{for 3 particular}\\ \text{beads grouped}\\ \text{as 2 units (2}\\ \text{together, 1 not),}\\ \text{and 2! ways to}\\ \text{arrange among}\\ \text{themselves}}} = 60480$$

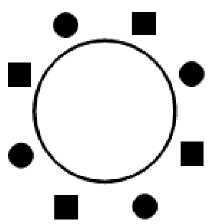
$\therefore$ total no. of ways $= 30240 + 60480 = 90720$

(iii)

If spherical beads and cubic beads alternate, then there must be 4 spherical beads and 4 cubic beads.

No. of ways

$$= \underbrace{{}^{5}C_4\,(4-1)!}_{\substack{\text{4 spherical beads and}\\ \text{arranged in a circle}}} \times \underbrace{{}^{5}C_4\,4!}_{\substack{\text{4 cubic beads,}\\ \text{and 4! ways to}\\ \text{arrange among}\\ \text{themselves}}} = 3600$$



Q2

(i)

<u>Method 1</u>: (using permutations)

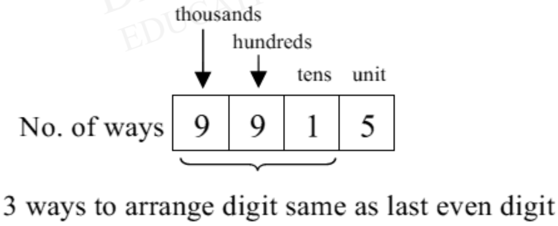
Probability $= \frac{10\times9\times8\times7}{10^4} = \frac{63}{125}$ [or 0.504]

<u>Method 2</u>: (using probability)

Probability $= \frac{10}{10}\times\frac{9}{10}\times\frac{8}{10}\times\frac{7}{10} = \frac{63}{125}$ [or 0.504]

(ii)

<u>Method 1</u>: (using permutations)

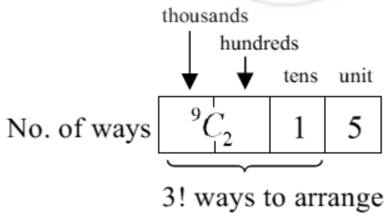


Required probability $= \frac{[(9\times9\times1)\times3]\times5}{10^4}$

$= \frac{243}{2000}$ [or 0.1215]

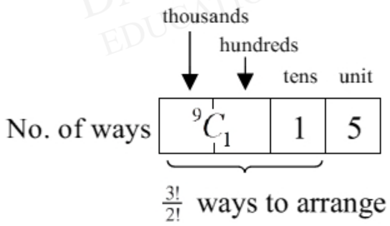
<u>Method 2</u>: (using permutations and combinations)

Case 1: The other 2 digits are different



Probability $= \frac{[(^9C_2\times1)\times3!]\times5}{10^4} = \frac{27}{250}$ [or 0.108]

Case 2: The other 2 digits are the same



Probability $= \frac{[(^9C_1\times1)\times\frac{3!}{2!}]\times5}{10^4} = \frac{27}{2000}$ [or 0.0135]

Required probability $= \frac{27}{250} + \frac{27}{2000} = \frac{243}{2000}$ [or 0.1215]

<u>Method 3</u>: (using probability)

Case 1: The other 2 digits are different

Probability $=\frac{9}{10}\times\frac{8}{10}\times\frac{1}{10}\times\frac{5}{10}\times\frac{3!}{2!}=\frac{27}{250}$ [or 0.108]

Case 2: The other 2 digits are the same

Probability $=\frac{9}{10}\times\frac{1}{10}\times\frac{1}{10}\times\frac{5}{10}\times\frac{3!}{2!}=\frac{27}{2000}$ [or 0.0135]

Required probability $=\frac{27}{250}+\frac{27}{2000}=\frac{243}{2000}$ [or 0.1215]

(iii)

Let A be the event '4 different digits with 1st digit greater than 6'.
Let B be the event 'odd and even digits that alternate'.

Method 1: (using permutations)

Case 1: 1st digit is even, i.e. 8, and odd and even digits alternate

	thousands	hundreds	tens	unit
No. of ways	1	5	4	4

Probability $=\frac{1\times5\times4\times4}{10^4}=\frac{1}{125}$ [or 0.008]

Case 2: 1st digit is odd, i.e. 7 or 9, and odd and even digits alternate

	thousands	hundreds	tens	unit
No. of ways	2	5	4	4

Probability $=\frac{\quad}{10^4}=\frac{2}{125}$ [or 0.016]

Hence $\mathrm{P}(A\cap B)=\frac{1}{125}+\frac{2}{125}=\frac{3}{125}$ [or 0.024]

$\mathrm{P}(B)=\mathrm{P}(\text{'odd,even,odd,even' or 'even,odd,even,odd'})$

$=\frac{2\times(5\times5\times5\times5)}{10^4}$

$=\frac{1}{8}$ [or 0.125]

$\therefore\ \mathrm{P}(A\mid B)=\frac{\mathrm{P}(A\cap B)}{\mathrm{P}(B)}=\frac{\frac{3}{125}}{\frac{1}{8}}=\frac{24}{125}$ [or 0.192]

Method 2: (using probability)

Case 1: 1st digit is even, i.e. 8, and odd and even digits alternate

Probability $=\frac{1}{10}\times\frac{5}{10}\times\frac{4}{10}\times\frac{4}{10}=0.008$

Case 2: 1st digit is odd, i.e. 7 or 9, and odd and even digits alternate

$$\text{Probability } = \frac{2}{10} \times \frac{5}{10} \times \frac{4}{10} \times \frac{4}{10} = 0.016$$

Hence $\mathrm{P}(A \cap B) = \frac{1}{125} + \frac{2}{125} = \frac{3}{125}$ [or 0.024]

$$\mathrm{P}(B) = \mathrm{P}(\text{'odd,even,odd,even' or 'even,odd,even,odd'})$$

$$= 2 \times \left(\frac{5}{10} \times \frac{5}{10} \times \frac{5}{10} \times \frac{5}{10} \right)$$

$$= 0.125$$

$\therefore \mathrm{P}(A \mid B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)} = \frac{\frac{3}{125}}{\frac{1}{8}} = \frac{24}{125}$ [or 0.192]

Q3

(i) Use 1, 2 and 3 to form 3-digit numbers

(a) $3! = 6$

(b) $3 \times 3 \times 3 = 27$

(c) Method 1 Consider the complement

Number of 3-digit numbers with all 3 digits the same (AAA) = 3

Required number = $27 - 3 = 24$

Method 2 Consider cases

Case 1 Each digit is used exactly once

Number of 3-digit numbers = 6 (from (i)(a))

Case 2 One digit is used twice (AAB)

Number of 3-digit numbers = ${}^{3}P_{2} \times \frac{3!}{2!} = 18$

(${}^{3}P_{2} = 3 \times 2$: 3 ways to select a digit to be used twice; 2 ways to select another digit)

Total number of 3-digit numbers = 6 + 18 = 24

(ii) Use 1, 2 and 3 to form 4-digit numbers

Method 1 Consider the complement

Total number of 4-digit numbers = $3^4 = 81$

Case 1 AAAB

Number of 4-digit numbers = ${}^{3}P_{2} \times \frac{4!}{3!} = 24$

(${}^{3}P_{2} = 3 \times 2$: 3 ways to select a digit to be used thrice; 2 ways to select another digit)

Case 2 AAAA

Number of 4-digit numbers = 3

Total number of 4-digit numbers = $81 - (24 + 3) = 54$

Method 2 Consider cases

Case 1 AABC

Number of 4-digit numbers = $3 \times \frac{4!}{2!} = 36$

(3 ways to select the digit to be used twice)

Case 2 AABB

Number of 4-digit numbers = ${}^{3}C_{2} \times \frac{4!}{2! \times 2!} = 18$

(${}^{3}C_{2}$ ways to select the 2 digits each to be used twice)

Total number of 4-digit numbers = 36 + 18 = 54