A Level H2 Math

Normal Distribution Test 2

Q1

A trading card game has rectangular cards of nominal size 64 mm wide and 89 mm long. However, due to the limited precision of the machine used to cut the cards to size, the widths of the trading cards follow a normal distribution with mean 64 mm and standard deviation 0.3 mm. The lengths of the trading cards follow an independent normal distribution with mean 89 mm and standard deviation 0.45 mm. The perimeter of the trading cards is twice the sum of its length and width.

(i) Trading cards with length 90 mm and above are called "tall" cards. Find the percentage of trading cards that are "tall". [1]

(ii) Write down the distribution of the perimeter of the trading cards, in mm, and find the perimeter that is exceeded by 8% of the trading cards. [4]

A brand of rectangular card sleeves are manufactured for the trading cards and the widths of the card sleeves follow a normal distribution with mean 66 mm and standard deviation 0.45 mm, whereas the lengths of the card sleeves follow an independent normal distribution with mean 91 mm and standard deviation 0.675 mm.

For a card sleeve to fit the trading card nicely, the dimensions of the sleeves must be larger than the dimensions of the trading card, but there should only be a maximum allowance of 1.2 mm on each side.

(iii) Find the probability that a randomly chosen card sleeve fits a randomly chosen trading card nicely, stating clearly the parameters of any distribution used. [5]

Q2

The number of days of gestation for a Dutch Belted cow is normally distributed, with a mean of μ days and a standard deviation of σ days. 8.08% of this cattle breed has a gestation period shorter than 278 days whereas 21.2% has a gestation period longer than 289 days. Find the values of μ and σ, giving your answers correct to 3 significant figures. [3]

(i) Find the probability that the mean gestation period for thirty-two randomly chosen Dutch Belted cows is more than 287 days. State a necessary assumption for your calculation to be valid. [3]

For another cattle breed, the Jersey cow, the number of days of gestation is normally distributed with a mean of 278 days and a standard deviation of 2.5 days.

During gestation, a randomly chosen pregnant Dutch Belted cow eats 29 kg of feed daily while a randomly chosen pregnant Jersey cow eats 26 kg of feed daily.

(ii) Find the value of a such that during their respective gestation periods, there is a probability of 0.35 that the amount of feed consumed by a randomly chosen pregnant Jersey cow exceeds half of the amount consumed by a randomly chosen pregnant Dutch Belted cow by less than a kg. Express your answer to the nearest kg. [2]

(iii) Calculate the probability that during their respective gestation periods, the difference between the amount of feed consumed by three randomly chosen pregnant Dutch Belted cows and four randomly chosen pregnant Jersey cows is more than 4000 kg. State clearly the parameters of the distribution used in the calculation. [3]

Q3

The students in a college are separated into two groups of comparable sizes, Group X and Group Y. The marks for their Mathematics examination are normally distributed with means and variances as shown in the following table.

	Mean	Variance
Group X	55	20
Group Y	34	25

(i) Explain why it may not be appropriate for the mark of a randomly chosen student from the college population to be modelled by a normal distribution. [1]

(ii) In order to pass the examination, students from Group Y must obtain at least d marks. Find, correct to 1 decimal place, the maximum value of d if at least 60% of them pass. [3]

(iii) Find the probability that the total marks of 4 students from Group Y is less than three times the mark of a student from Group X. State clearly the mean and variance of the distribution you use in your calculation. [3]

(iv) The marks of 40 students, with 20 each randomly selected from Group X and Group Y, are used to compute a new mean mark , $\bar{M}$. Given that $\mathrm{P}(|\bar{M}-44.5|<k)=0.9545$, find the value of k. [4]

State a necessary assumption for your calculations to hold in parts **(iii)** and **(iv)**. [1]

Answers

Normal Distribution Test 2

Q1

(i)	Let L be the random variable denoting length of a trading card in mm. $L \sim N(89, 0.2025)$ $P(L > 90) = 0.0131$, hence the percentage is 1.31%.	Badly done by students. Mistakes: 1) Confusing CRV and DRV by writing $P(L \le 90) = P(L \le 89)$ 2) Not answering in %
(ii)	Let T be the random variable denoting the perimeter of a trading card, in mm. $T \sim N\left(2(64)+2(89),\ 2^2(0.3^2)+2^2(0.45^2)\right)$ $\sim N(306,\ 1.17)$ $P(T > t) = 0.08$ $P(T < t) = 0.92$ Hence $t = 307.51\text{mm}$	Badly done by students. Most common mistake: Taking invNorm with RHS area 0.2.
(iii)	Let X and Y be random variable denoting the width and length of a card sleeve subtracting away the width and length of a trading card respectively in mm. Hence $X \sim N\left(66-64,\ (0.45^2)+(0.3^2)\right)$ $\sim N(2,\ 0.2925)$ and $Y \sim N\left(91-89,\ (0.45^2)+(0.675^2)\right)$ $\sim N(2,\ 0.658125)$ $\text{P(width fits nicely)} = P(0 < X \le 2.4) = 0.7701200999$ $\text{P(length fits nicely)} = P(0 < Y \le 2.4) = 0.6821730404$ $\text{P(sleeve fits nicely)} = \text{P(width fits nicely)}\,\text{P(length fits nicely)}$ $= 0.7701200999 \times 0.6821730404$ $= 0.525$	Badly done. Better students were able to find the new mean and new variance but mistakes were made when calculate the probabilities, Mistakes: 1) $P(X \le 2.4)P(Y \le 2.4)$ 2) $P(X \le 1.2)P(Y \le 1.2)$ 3) $P(0 \le X \le 1.2)P(0 \le Y \le 1.2)$ Most students left this part blank.

Q2

Let D be the random variable denoting the number of days of gestation for a Dutch Belted cow.

$$\mathrm{P}(D < 278) = 0.0808$$

$$\mathrm{P}(Z < \frac{278-\mu}{\sigma}) = 0.0808$$

$$\frac{278-\mu}{\sigma} = -1.39971 --- (1)$$

$$\mathrm{P}(D > 289) = 0.212$$

$$\mathrm{P}(Z < \frac{289-\mu}{\sigma}) = 0.788$$

$$\frac{289-\mu}{\sigma} = 0.799501 --- (2)$$

Solving (1)&(2), $\mu = 285.001064$ and $\sigma = 5.0017961$

$\mu = 285$ (3 s.f.) and $\sigma = 5.00$ (3 s.f.).

(i)

$$\overline{D} \sim \mathrm{N}(285.001064, \frac{5.0017961^2}{32}).$$

$$\mathrm{P}(\overline{D} > 287) = 0.0118629 = 0.0119 \text{ (3 s.f.)}$$

The <u>number of days of gestation for a Dutch Belted cow</u> is <u>independent</u> of <u>the number of days of gestation of another Dutch Belted cow.</u>

(ii)

$$J \sim \mathrm{N}(278, 2.50^2)$$

$$D \sim \mathrm{N}(285.001064, 5.0017961^2)$$

Let $X = 26J - \frac{1}{2}29D$

$$X \sim \mathrm{N}(3095.48457 \text{ , } 9485.02698)$$

$$\mathrm{P}(0 < X < a) = 0.35$$

$$\therefore a = 3057.95778 \approx 3058$$

(iii)

$D \sim \mathrm{N}(285.001064, 5.0017961^2)$

$J \sim \mathrm{N}(278, 2.50^2)$

Let C_1 denote the random variable of the amount of feed cosumed by 3 pregnant Dutch belted cows.

Let C_2 denote the random variable of the amount of feed cosumed by 4 pregnant Jersey cows.

$C_1 = 29(D_1 + D_2 + D_3) \sim \mathrm{N}(24795.09257, 63120.32374)$

$C_2 = 26(\mathrm{J}_1 + J_2 + J_3 + J_4) \sim \mathrm{N}(28912, 16900)$

$C_1 - C_2 \sim \mathrm{N}(-4117, 80020.3237)$

$\mathrm{P}(|C_1 - C_2| > 4000)$

$= \mathrm{P}(C_1 - C_2 < -4000) + \mathrm{P}(C_1 - C_2 > 4000)$

$= 0.6604182314$

$= 0.660$ (3 s.f.)

Or

$D \sim \mathrm{N}(285.00, 5.0018^2)$

$J \sim \mathrm{N}(278, 2.50^2)$

Let C_1 denote the random variable of the amount of feed cosumed by 3 pregnant Dutch belted cows.

Let C_2 denote the random variable of the amount of feed cosumed by 4 pregnant Jersey cows.

$C_1 = 29(D_1 + D_2 + D_3) \sim \mathrm{N}(24795, 63120.42217)$

$C_2 = 26(\mathrm{J}_1 + J_2 + J_3 + J_4) \sim \mathrm{N}(28912, 16900)$

$C_1 - C_2 \sim \mathrm{N}(-4117, 80020.422)$

$\mathrm{P}(|C_1 - C_2| > 4000)$

$= \mathrm{P}(C_1 - C_2 < -4000) + \mathrm{P}(C_1 - C_2 > 4000)$

$= 0.66041814$

$= 0.660$ (3 s.f.)

Q3

(i) The distribution may become <u>bimodal</u> when the data for both groups are combined

(ii) Let Y be the score of a random student from Group Y. $Y \sim \text{N}(34,\ 25)$

$\text{P}(Y \geq d) \geq 0.6$

$\text{P}(Y \leq d) \leq 0.4$

When $\text{P}(Y \leq d_c) = 0.4$, $d_c = 32.733$.

Thus $d < 32.733$. The maximum mark is 32.7

(iii)
$$\text{E}\left(\sum_1^4 Y_i - 3X\right) = 4\text{E}(Y) - 3\text{E}(X) = -29$$

$$\text{Var}\left(\sum_1^4 Y_i - 3X\right) = 4\text{Var}(Y) + 9\text{Var}(X) = 280$$

$$\therefore \sum_1^4 Y_i - 3X \sim \text{N}(-29, 280)$$

$$\text{P}(\sum_1^4 Y_i < 3X) = \text{P}(\sum_1^4 Y_i - 3X < 0) = 0.958 \text{ (to 3sf)}$$

(iv)
$$\overline{M} = \frac{\sum_{i=1}^{20} X_i + \sum_{i=1}^{20} Y_i}{40}$$

$$\text{E}\left(\overline{M}\right) = \frac{20\text{E}(X) + 20\text{E}(Y)}{40} = \frac{1}{2}\left(\text{E}(X) + \text{E}(Y)\right) = 44.5$$

Let $\sigma^2 = \text{Var}\left(\overline{M}\right)$

$$= \frac{1}{1600}\left(20\text{Var}(X) + 20\text{Var}(Y)\right)$$

$$= \frac{1}{80}\left(\text{Var}(X) + \text{Var}(Y)\right) = 0.5625$$

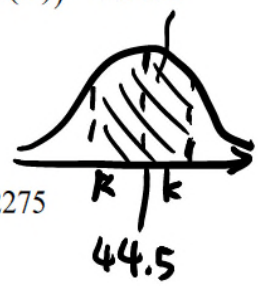
$\overline{M} \sim N(44.5, 0.5625)$

Since $\text{P}(|\overline{M} - 44.5| < k) = 0.9545$

$\Rightarrow \text{P}(\overline{M} < 44.5 - k) = \frac{1 - 0.9545}{2} = 0.02275$

$\therefore 44.5 - k = 43.000$

$\Rightarrow k = 1.50$ (3 s.f)



<u>Alternative</u>

$\overline{M} \sim N(44.5, \sigma^2)$

Since $\text{P}(|\overline{M} - 44.5| < 2\sigma) = 0.9545$

$\therefore k = 2\sigma = 2\sqrt{0.5625} = 1.50$ (3sf)

<u>*Marks of students*</u> are independent of one another.