A Level H2 Math

Maclaurin Series Test 7

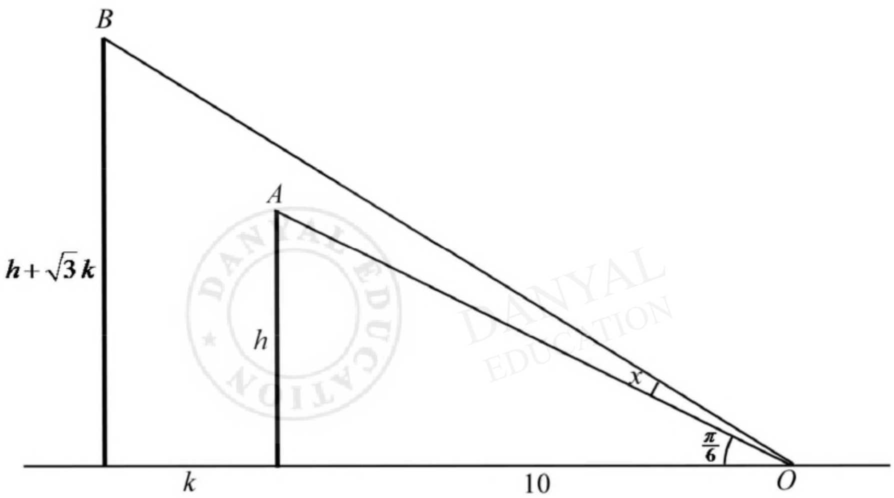
Q1

(i) Expand $(k+x)^n$, in ascending powers of x, up to and including the term in x^2, where k is a non-zero real constant and n is a negative integer. [3]

(ii) State the range of values of x for which the expansion is valid. [1]

(iii) In the expansion of $(k+y+3y^2)^{-3}$, the coefficient of y^2 is 2. By using the expansion in **(i)**, find the value of k. [3]

Q2



A laser from a fixed point O on a flat ground projects light beams to the top of two vertical structures A and B as shown above. To project the beam to the top of A, the laser makes an angle of elevation of $\frac{\pi}{6}$ radians. To project the beam to the top of B, the laser makes an angle of elevation of $\left(\frac{\pi}{6}+x\right)$ radians. The two structures A and B are of heights h m and $\left(h+\sqrt{3}k\right)$ m respectively and are 10 m and $(10+k)$ m away from O respectively.

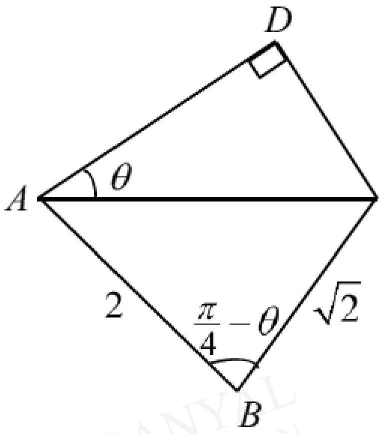
(i) Show that the length of the straight beam from O to A is $\frac{20}{\sqrt{3}}$ m. [1]

(ii) Show that the length of AB is $2k$ m and that the angle of elevation of B from A is $\frac{\pi}{3}$ radians. [3]

(iii) Hence, using the sine rule, show that $k=\dfrac{10\sin x}{\sqrt{3}\sin\left(\frac{\pi}{6}-x\right)}$. [2]

(iv) If x is sufficiently small, show that $k \approx \frac{20}{\sqrt{3}}\left(x+ax^2\right)$, where a is a constant to be determined. [6]

Q3



The diagram above shows a quadrilateral *ABCD*, where $AB = 2$, $BC = \sqrt{2}$, angle $ABC = \dfrac{\pi}{4} - \theta$ radians and angle $CAD = \theta$ radians.

Show that

$$AC = \sqrt{6 - 4\cos\theta - 4\sin\theta}. \qquad [2]$$

Given that θ is small enough for θ^3 and higher powers of θ to be neglected, show that

$$AD \approx a + b\theta + c\theta^2,$$

where *a*, *b* and *c* are constants to be determined. [5]

Answers

Binomial Expansion Test 1

Q1

(i)

$$(k+x)^n = k^n\left(1+\frac{x}{k}\right)^n$$
$$= k^n\left(1+n\left(\frac{x}{k}\right)+\frac{(n)(n-1)}{2!}\left(\frac{x}{k}\right)^2+\dots\right)$$
$$= k^n\left(1+\frac{n}{k}x+\frac{(n)(n-1)}{2k^2}x^2+\dots\right)$$

(i) Most candidates knew more or less what to do, although mistakes were common; the most common were $(k+x)^n = k\left(1+\frac{x}{k}\right)^n$ or

$$(k+x)^n = \left(\frac{1}{k}\right)^n(1+kx)^n$$
$$= \left(\frac{1}{k}\right)^n\left(1+nkx+\frac{(n)(n-1)}{2!}(kx)^2+\dots\right)$$

Some left answer as

$$(k+x)^n = k^n\left(1+n\left(\frac{x}{k}\right)+\frac{(n)(n-1)}{2!}\left(\frac{x}{k}\right)^2+\dots\right.$$

Did not simplify $\left(\frac{x}{k}\right)^2 = \frac{x^2}{k^2}$

No marks was awarded for

$$(k+x)^n \cong \left(k^n+nk^{n-1}x+\frac{(n)(n-1)}{2}k^{n-2}x^2\right).$$

And

$$(k+x)^n = \left(k^n+\binom{n}{1}k^{n-1}x+\binom{n}{2}k^{n-2}x^2+\dots\dots\right)$$
$$= \left(k^n+nk^{n-1}x+\frac{(n)(n-1)}{2}k^{n-2}x^2\right)$$

(ii)

$$\left|\frac{x}{k}\right|<1 \Rightarrow |x|<|k|$$
$$\therefore -|k|<x<|k|$$

Very badly done . Do not know how to proceed after $\left|\frac{x}{k}\right|<1$ and left answers like $|x|<|k|$ or $-k<x<k$ or $-1<x<1$

Candidates who used Maclaurin series to find the binomial expansion of $(k+x)^n$ have problems finding region of validity. Gave answers like $|x|<1$ or $x\in R$

(iii)

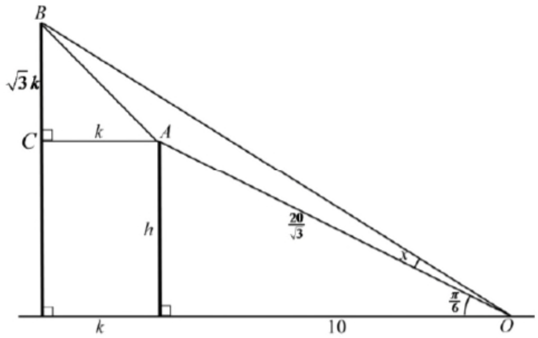
Let $x = y+3y^2$ and $n=-3$:

$$\left(k+y+3y^2\right)^{-3}$$
$$= k^{-3}\left(1+\frac{(-3)}{k}\left(y+3y^2\right)+\frac{(-3)(-4)}{2k^2}\left(y+3y^2\right)^2+\dots\right)$$
$$= k^{-3}\left(1-\frac{3}{k}y-\frac{9}{k}y^2+\frac{6}{k^2}y^2+\dots\right)$$
$$\Rightarrow k^{-3}\left(-\frac{9}{k}+\frac{6}{k^2}\right)=2 \Rightarrow 2k^5+9k-6=0$$
$$\therefore k = 0.642 \text{ (to 3 sf)}$$

Surprisingly quite a number of students do not know how to solve $-\frac{9}{k^4}+\frac{6}{k^5}=2$ or $2k^5+9k-6=0$

Q2

(i)



$$\cos\frac{\pi}{6} = \frac{10}{OA}$$
$$\frac{\sqrt{3}}{2} = \frac{10}{OA}$$
$$OA = \frac{20}{\sqrt{3}}\text{ m}$$

(Shown)

(ii)

$$AB = \sqrt{k^2+3k^2} = \sqrt{4k^2} = 2k \qquad \text{(Shown)}$$
$$\angle BAC = \tan^{-1}\frac{\sqrt{3}\,k}{k} = \tan^{-1}\sqrt{3}$$
$$= \frac{\pi}{3} \qquad \text{(Shown)}$$

(iii)

$$\angle CBO = \tfrac{\pi}{2} - \left(\tfrac{\pi}{6}+x\right) = \tfrac{\pi}{3} - x$$
$$\angle CBA = \tfrac{\pi}{2} - \tfrac{\pi}{3} = \tfrac{\pi}{6}$$
$$\angle ABO = \tfrac{\pi}{3} - x - \tfrac{\pi}{6} = \tfrac{\pi}{6} - x$$

Or:

$$\angle BAO = 2\pi - \tfrac{\pi}{2} - \tfrac{\pi}{3} - \tfrac{\pi}{3} \quad (\angle \text{ at a pt})$$
$$= \tfrac{5\pi}{6}$$
$$\angle ABO = \pi - x - \tfrac{5\pi}{6} = \tfrac{\pi}{6} - x$$

In ΔABO,

$$\frac{2k}{\sin x} = \frac{\frac{20}{\sqrt{3}}}{\sin\left(\frac{\pi}{6}-x\right)}$$
$$k = \frac{10\sin x}{\sqrt{3}\sin\left(\frac{\pi}{6}-x\right)}$$

(iv)

$$k = \frac{10\sin x}{\sqrt{3}\sin\left(\frac{\pi}{6}-x\right)}$$
$$= \frac{10\sin x}{\sqrt{3}\left(\sin\frac{\pi}{6}\cos x - \cos\frac{\pi}{6}\sin x\right)}$$
$$\approx \frac{10x}{\sqrt{3}\left[\frac{1}{2}\left(1-\frac{x^2}{2}\right) - \frac{\sqrt{3}}{2}x\right]}$$
$$= \frac{10x}{\frac{\sqrt{3}}{2}\left[\left(1-\frac{x^2}{2}\right) - \sqrt{3}\,x\right]}$$
$$= \frac{20x}{\sqrt{3}}\left[1-\left(\sqrt{3}x+\frac{x^2}{2}\right)\right]^{-1}$$
$$\approx \frac{20x}{\sqrt{3}}\left(1+\sqrt{3}\,x\right)$$
$$= \underline{\underline{\frac{20}{\sqrt{3}}\left(x+\sqrt{3}\,x^2\right)}}$$

Q3

Consider triangle ABC,

$$AC^2 = 4+2-2(2)\sqrt{2}\cos\left(\frac{\pi}{4}-\theta\right)$$

$$= 6-4\sqrt{2}\left(\cos\frac{\pi}{4}\cos\theta+\sin\frac{\pi}{4}\sin\theta\right) = 6-4\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)$$

$$AC = \sqrt{6-4\cos\theta-4\sin\theta} \text{ (shown)}$$

Consider triangle ACD,

$$\cos\theta = \frac{AD}{AC}$$

$$AD = \cos\theta\sqrt{6-4\cos\theta-4\sin\theta}$$

Since θ is small, $\sin\theta\approx\theta,\ \cos\theta\approx 1-\frac{\theta^2}{2}$,

$$AD \approx \left(1-\frac{\theta^2}{2}\right)\sqrt{6-4\left(1-\frac{\theta^2}{2}\right)-4\theta}$$

$$= \left(1-\frac{\theta^2}{2}\right)\left(2+2\theta^2-4\theta\right)^{\frac{1}{2}}$$

$$= \left(1-\frac{\theta^2}{2}\right)\sqrt{2}\left(1+\theta^2-2\theta\right)^{\frac{1}{2}}$$

$$= \sqrt{2}\left(1-\frac{\theta^2}{2}\right)\left(1+\frac{1}{2}(\theta^2-2\theta)+\frac{(\frac{1}{2})(-\frac{1}{2})}{2}(\theta^2-2\theta)^2+\ldots\right)$$

$$= \sqrt{2}\left(1-\frac{\theta^2}{2}\right)\left(1+\frac{1}{2}\theta^2-\theta-\frac{1}{2}\theta^2+\ldots\right)$$

$$= \sqrt{2}\left(1-\frac{\theta^2}{2}\right)(1-\theta+\ldots)$$

$$= \sqrt{2}\left(1-\theta-\frac{\theta^2}{2}+\ldots\right)$$

$$\approx \sqrt{2}-\sqrt{2}\theta-\frac{\sqrt{2}}{2}\theta^2$$