A Level H2 Math

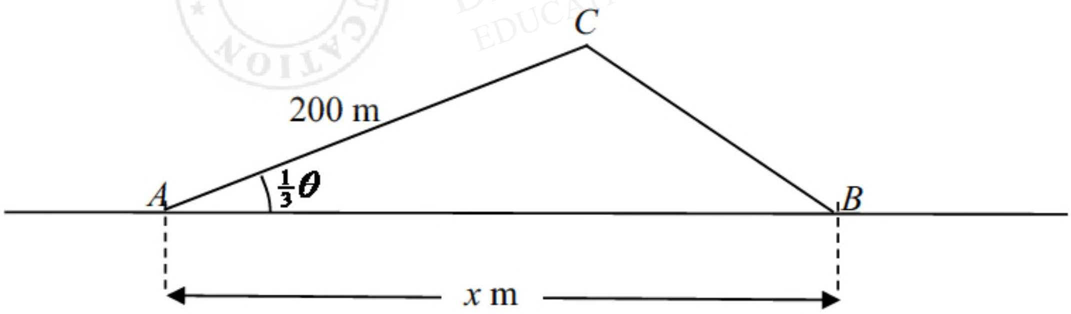
Maclaurin Series Test 6

Q1

(i) Given that $\mathrm{f}(x)=\tan\left(\frac{1}{2}x+\frac{1}{4}\pi\right)$, show that $\mathrm{f}'(x)=\frac{1}{2}\left[1+\left(\mathrm{f}(x)\right)^2\right]$, and find $\mathrm{f}(0)$, $\mathrm{f}'(0)$, $\mathrm{f}''(0)$ and $\mathrm{f}'''(0)$. Hence write down the first four non-zero terms in the Maclaurin series for $\mathrm{f}(x)$. [7]

(ii) The first three non-zero terms in the Maclaurin series for $\mathrm{f}(x)$ are equal to the first three non-zero terms in the series expansion of $\frac{\cos(ax)}{1+bx}$. By using appropriate expansions from the List of Formulae (MF26), find the possible value(s) for the constants a and b. [5]

Q2



The diagram shows two points at ground level, A and B. The distance in metres between A and B is denoted by x. The angle of elevation of C from B is twice the angle of elevation of C from A. The distance AC is 200 m and $\angle BAC=\frac{1}{3}\theta$ radians. Show that

$$x=\frac{200\sin\theta}{\sin\frac{2}{3}\theta}.$$ [2]

It is given that θ is a small angle such that θ^4 and higher powers of θ are negligible. By using appropriate expansions from the List of Formulae (MF26), show that

$$x\approx\frac{2700-250\theta^2}{9}.$$ [4]

Q3

Given that $e^y = \sqrt{e + x + \sin x}$. Show that

$$2e^{2y}\frac{d^2y}{dx^2} + 4e^{2y}\left(\frac{dy}{dx}\right)^2 + \sin x = 0 .$$ [2]

(i) Find the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ when $x = 0$. Hence, find in terms of e, the Maclaurin's series for $\ln(e + x + \sin x)$, up to and including the term in x^2. [4]

(ii) By using appropriate standard series expansions from the List of Formulae (MF26), verify the correctness of the first three terms in the series expansion for $\ln(e + x + \sin x)$ found in part **(i)**. [3]

(iii) Use your answer to part **(i)** to give an approximation for $\int_0^{e^{-1}} \frac{2e - 4x}{e^2 \ln(e + x + \sin x)}\, dx$, giving your answer in terms of e. [3]

Answers
Maclaurin Series Test 6

Q1

(i) $\mathrm{f}(x)=\tan\left(\frac{1}{2}x+\frac{1}{4}\pi\right)$

$$\mathrm{f}'(x)=\frac{1}{2}\sec^2\left(\frac{1}{2}x+\frac{1}{4}\pi\right)$$
$$=\frac{1}{2}\left[1+\tan^2\left(\frac{1}{2}x+\frac{1}{4}\pi\right)\right]$$
$$=\frac{1}{2}\left(1+\left(\mathrm{f}(x)\right)^2\right) \qquad \text{(shown)}$$
$$\mathrm{f}''(x)=\mathrm{f}(x)\mathrm{f}'(x)$$
$$\mathrm{f}'''(x)=\mathrm{f}(x)\mathrm{f}''(x)+\left(\mathrm{f}'(x)\right)^2$$
$\mathrm{f}(0)=1$,
$\mathrm{f}'(0)=1$,
$\mathrm{f}''(0)=1$,
$\mathrm{f}'''(0)=2$.
$$\therefore \quad \mathrm{f}(x)=1+x+\frac{1}{2}x^2+\frac{1}{3}x^3\cdots$$

(ii)
$$\frac{\cos(ax)}{1+bx}=\left(1-\frac{(ax)^2}{2!}+\ldots\right)\left(1+(-1)(bx)+\frac{(-1)(-2)}{2!}(bx)^2+\ldots\right)$$
$$=\left(1-\frac{a^2x^2}{2}+\ldots\right)\left(1-bx+b^2x^2+\ldots\right)$$
$$\approx 1-bx+b^2x^2-\frac{a^2x^2}{2}$$
$$=1-bx+\left(b^2-\frac{a^2}{2}\right)x^2$$

Comparing coefficients,
$x:b=-1$
$$x^2:b^2-\frac{a^2}{2}=\frac{1}{2}\Rightarrow\frac{a^2}{2}=\frac{1}{2}\Rightarrow a=\pm 1$$

Q2

$\angle ABC = 2\times\angle BAC = \dfrac{2\theta}{3}$ $\Rightarrow \quad \angle ACB = \pi - \theta$ Using Sine rule, $\dfrac{x}{\sin(\pi-\theta)} = \dfrac{200}{\sin\left(\frac{2\theta}{3}\right)}$ Since $\sin(\pi-\theta) = \sin\pi\cos\theta - \cos\pi\sin\theta = \sin\theta$, $\dfrac{x}{\sin\theta} = \dfrac{200}{\sin\left(\frac{2\theta}{3}\right)}$ $\Rightarrow \quad x = \dfrac{200\sin\theta}{\sin\left(\frac{2}{3}\theta\right)}$ (Shown)	A common mistake is $\angle ACB = 2\theta - \dfrac{\theta}{3} - \dfrac{2\theta}{3} = \theta$. Students who made this mistake simply wanted θ to appear and do not ensure that the expression is true. ***Note:*** This is a “Show” question. Thus **<u>all</u>** working/explanation **<u>should</u>** be clearly stated, i.e., need to show $\angle ACB$ and $\angle ABC$, and state the method (sine rule) used.

$x = \dfrac{200\sin\theta}{\sin\left(\frac{2}{3}\theta\right)} \approx \dfrac{200\left(\theta - \frac{\theta^3}{3!}\right)}{\left(\frac{2}{3}\theta - \frac{\left(\frac{2}{3}\theta\right)^3}{3!}\right)}$ since θ^4 and higher powers of θ are negligible	Note that since “+…” is dropped, the “$\approx$” sign should be used.
$= \dfrac{200\,\theta\left(1-\frac{\theta^2}{6}\right)}{\frac{2}{3}\theta\left(1-\frac{2\theta^2}{27}\right)}$	Take out θ and cancel for easy computation.
$= 300\left(1-\dfrac{\theta^2}{6}\right)\left(1-\dfrac{2\theta^2}{27}\right)^{-1}$ $= 300\left(1-\dfrac{\theta^2}{6}\right)\left(1+\dfrac{2\theta^2}{27}+\ldots\right)$ $= 300\left(1-\dfrac{5\theta^2}{54}+\ldots\right)$ $\approx \dfrac{2700-250\theta^2}{9}$	To ensure that the final expression is a polynomial, the denominator has to be “brought up” using power -1. Then use the expansion $(1+x)^{-1}$.

Marker’s comments

For the 1st part, many students attempted to find x by considering the two triangles formed by drawing a line through C perpendicular to AB. This method is tedious.

The $\sin\theta$ and $\sin\left(\dfrac{2}{3}\theta\right)$ in the expression to be shown would suggest using sine rule.

Q3

$e^y = \sqrt{e + x + \sin x}$ $\Rightarrow \quad e^{2y} = e + x + \sin x$ Differentiate wrt x: $e^{2y}\left(2\frac{dy}{dx}\right) = 1 + \cos x$ i.e., $2e^{2y}\frac{dy}{dx} = 1 + \cos x$ Differentiate wrt x: $2\left[e^{2y}\frac{d^2y}{dx^2} + \frac{dy}{dx}e^{2y}\left(2\frac{dy}{dx}\right)\right] = -\sin x$ i.e., $2e^{2y}\frac{d^2y}{dx^2} + 4e^{2y}\left(\frac{dy}{dx}\right)^2 + \sin x = 0$ (Shown)	Square both sides first. Do not use the tedious method of working out $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ directly from given equation.

(i) When $x = 0$, $e^{2y} = e + 0 + 0 \Rightarrow y = \frac{1}{2}$ $2e\frac{dy}{dx} = 1 + 1 \Rightarrow \frac{dy}{dx} = \frac{1}{e}$ $2e\frac{d^2y}{dx^2} + 4e\left(\frac{1}{e}\right)^2 + 0 = 0 \Rightarrow \frac{d^2y}{dx^2} = \frac{-2}{e^2}$ $\therefore \quad y = \frac{1}{2} + \frac{1}{e}x - \frac{2}{e^2}\left(\frac{x^2}{2!}\right) + \cdots$ $e^{2y} = e + x + \sin x$ $\Rightarrow \ln(e + x + \sin x) = 2y$ $= 2\left(\frac{1}{2} + \frac{1}{e}x - \frac{2}{e^2}\frac{x^2}{2!} + \cdots\right)$ i.e., $\ln(e + x + \sin x) = 1 + \frac{2}{e}x - \frac{2}{e^2}x^2 + \cdots$	This series expansions is for y, and **not** for $\ln(e+x+\sin x)$ or e^{2y}.
(ii) $\ln(e + x + \sin x)$ $= \ln(e + x + x + \ldots)$ $= \ln(e + 2x + \ldots)$ $= \ln\left(e\left(1 + \frac{2}{e}x + \ldots\right)\right)$ $= \ln e + \ln\left(1 + \frac{2}{e}x + \ldots\right)$ $= 1 + \ln\left(1 + \frac{2}{e}x + \ldots\right)$ $= 1 + \left[\left(\frac{2}{e}x\right) - \frac{1}{2}\left(\frac{2}{e}x\right)^2 + \cdots\right]$ $= 1 + \frac{2}{e}x - \frac{2}{e^2}x^2 + \cdots$ (Verified)	Apply the following standard series expansions. $\sin x = x + \ldots$ and $\ln(1+x) = x - \frac{x^2}{2} + \ldots$. ($x^3$ term can be ignored as the result in part (i) is only up to x^2 term). *Note:* $\ln(e + x + \sin x) \neq \ln e + \ln x + \ln \sin x$ $\ln(e + x + \sin x) \neq (\ln e)\ln(x + \sin x)$, $\ln(e + x + \sin x) \neq \ln\left(1 + \frac{x}{e} + \frac{\sin x}{e}\right)$ You can use this (ii) result to check whether you make mistakes in part (i) or (ii) if the results are different.

(iii) $\int_0^{e^{-1}} \frac{2e-4x}{e^2 \ln(e+x+\sin x)}\,dx$ $\approx \int_0^{e^{-1}} \frac{2e-4x}{e^2\left(1+\frac{2}{e}x-\frac{2}{e^2}x^2\right)}\,dx$ $= \int_0^{e^{-1}} \frac{2e-4x}{e^2+2ex-2x^2}\,dx$ $= \left[\ln\left(e^2+2ex-2x^2\right)\right]_0^{\frac{1}{e}}$ $= \ln\left(e^2+2-2\left(\frac{1}{e}\right)^2\right)-\ln\left(e^2\right)$ $= \ln\left(e^2+2-\frac{2}{e^2}\right)-2$ $= \ln\left(e^4+2e^2-2\right)-4$	Use $\int \frac{f'(x)}{f(x)}\,dx = \ln\lvert f(x)\rvert + c$

Marker's comments

About 20% of the students use tedious method to show first part.

Badly done for part (ii). Many students leave blank for this part. For those who tried, many of them get different answers for (i) and (ii), and still wrote (verified). They should use it to check their own mistakes.