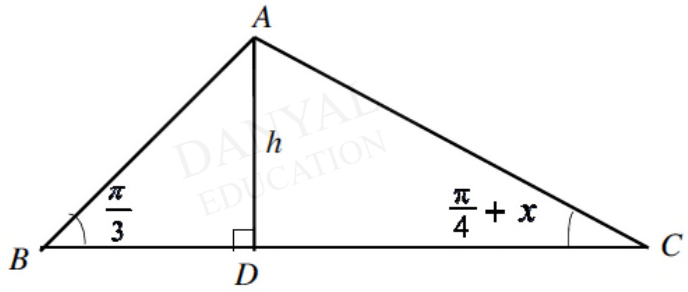
A Level H2 Math

Maclaurin Series Test 5

Q1

The diagram shows the triangle ABC. It is given that the height AD is h units, $\angle ABD = \frac{\pi}{3}$ and $\angle ACD = \frac{\pi}{4} + x$.



Show that if x is sufficiently small for x^3 and higher powers of x to be neglected, then

$$BC = \frac{h}{\sqrt{3}} + \frac{h}{\tan\left(\frac{\pi}{4} + x\right)} \approx h\left(p + qx + rx^2\right),$$

for constants p, q, r to be determined in exact form. **[5]**

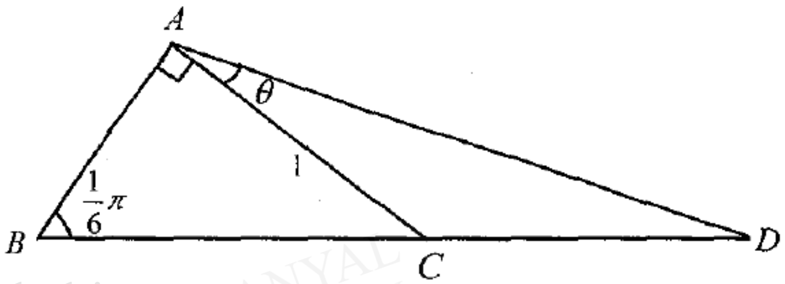
Q2

(a) It is given that $y=\mathrm{f}(x)$ is such that $my^2\dfrac{\mathrm{d}y}{\mathrm{d}x}-y^3=-\mathrm{e}^x\sin x$ and that the Maclaurin series for $\mathrm{f}(x)$ is given by $1+\dfrac{1}{3}x+nx^2+\ldots$, where m and n are some real constants.

(i) State the values of $\mathrm{f}(0)$ and $\mathrm{f}'(0)$. [2]

(ii) Find the values of m and n. [3]

(b) In the triangle ABC, $AC=1$, angle $BAC=\dfrac{\pi}{2}$ radians and angle $ABC=\dfrac{\pi}{6}$ radians. D is a point on BC produced such that angle $CAD=\theta$ radians (see diagram).



(i) Show that $AD=\dfrac{\sqrt{3}}{\sqrt{3}\cos\theta-\sin\theta}$. [4]

(ii) Given that θ is a sufficiently small angle, show that

$$AD\approx 1+a\theta+b\theta^2\ ,$$

for constants a and b to be determined exactly. [3]

Q3

It is given that $y = \ln(\cos ax - \sin ax)$, where a is a non-zero constant.

(i) Show that $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 + a^2 = 0$. [3]

(ii) By further differentiation of the result in **(i)**, find, in terms of a, the Maclaurin series for y, up to and including the term in x^3. [3]

(iii) Hence show that when x is small enough for powers of x higher than 2 to be neglected and $a = 2$, then $\cos 2x - \sin 2x \approx 1 + kx + kx^2$ where k is a constant to be determined. [4]

(iv) Using appropriate expansions from the List of Formulae (**MF26**), verify the correctness of your answer in **(iii)**. [2]

Answers
Maclaurin Series Test 5

Q1

(i)

$$BC = BD + DC$$

$$= \frac{h}{\tan\frac{\pi}{3}} + \frac{h}{\tan\left(\frac{\pi}{4}+x\right)}$$

(ii)

$$BC$$

$$= \frac{h}{\sqrt{3}} + \frac{h}{\frac{\tan\frac{\pi}{4}+\tan x}{1-\tan\frac{\pi}{4}\tan x}}$$

$$= \frac{h\sqrt{3}}{3} + \frac{h(1-\tan x)}{1+\tan x}$$

$$\approx \frac{h\sqrt{3}}{3} + \frac{h(1-x)}{1+x}$$

$$= \frac{h\sqrt{3}}{3} + h(1-x)(1+x)^{-1}$$

$$= \frac{h\sqrt{3}}{3} + h(1-x)[1+(-1)x+\frac{(-1)(-2)}{2!})x^2+...]$$

$$= \frac{h\sqrt{3}}{3} + h(1-x)[1-x+x^2+...]$$

$$= \frac{h\sqrt{3}}{3} + h(1-2x+2x^2+...)$$

$$= h\left(1+\frac{\sqrt{3}}{3} - 2x + 2x^2\right)$$

Q2

ai

$\mathrm{f}(x)=1+\frac{1}{3}x+nx^2+\ldots$

Comparing with

$\mathrm{f}(x)=\mathrm{f}(0)+\mathrm{f}'(0)x+\frac{\mathrm{f}''(0)}{2!}x^2+\ldots$

$\Rightarrow \mathrm{f}(0)=1$

$\Rightarrow \mathrm{f}'(0)=\frac{1}{3}$

aii

Given $my^2\frac{\mathrm{d}y}{\mathrm{d}x}-y^3=-\mathrm{e}^x\sin x$ ---(2)

When $x=0$,

$m(1)^2\left(\frac{1}{3}\right)-(1)^3=-\mathrm{e}^0\sin 0$

$\frac{1}{3}m=1$

$m=3$

Differentiate (2) w.r.t. x:

$3y^2\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+6\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-3y^2\frac{\mathrm{d}y}{\mathrm{d}x}=-\mathrm{e}^x\sin x-\mathrm{e}^x\cos x$

When $x=0$,

$3(1)^2(2n)+6\left(\frac{1}{3}\right)^2-3(1)^2\left(\frac{1}{3}\right)=-1$

$6n=-\frac{2}{9}$

$n=-\frac{1}{9}$

bi

Method 1

$\measuredangle ACD=\frac{\pi}{2}+\frac{\pi}{6}=\frac{2\pi}{3}$ (ext angle of a triangle)

Using Sine Rule in $\triangle ACD$

$\frac{AD}{\sin\frac{2\pi}{3}}=\frac{AC}{\sin\left(\pi-\frac{2\pi}{3}-\theta\right)}$

$AD=\frac{\sqrt{3}/2}{\sin\left(\frac{\pi}{3}-\theta\right)}$

$$= \frac{\sqrt{3}/2}{\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta}$$

$$= \frac{\sqrt{3}/2}{\left(\sqrt{3}/2\right)\cos\theta - \left(1/2\right)\sin\theta}$$

$$= \frac{\sqrt{3}}{\sqrt{3}\cos\theta - \sin\theta}$$

Method 2

In right-angled $\triangle ABC$, $AB = \dfrac{1}{\tan\frac{\pi}{6}} = \sqrt{3}$.

$\measuredangle ADB = \pi - \dfrac{\pi}{6} - \left(\dfrac{\pi}{2} + \theta\right) = \dfrac{\pi}{3} - \theta$ (angle sum of a triangle)

Using Sine Rule in $\triangle ABD$

$$\frac{AD}{\sin\frac{\pi}{6}} = \frac{AB}{\sin\left(\frac{\pi}{3} - \theta\right)}$$

$$AD = \frac{\sqrt{3}\sin\frac{\pi}{6}}{\sin\left(\frac{\pi}{3} - \theta\right)}$$

$$= \frac{\sqrt{3}/2}{\sin\frac{\pi}{3}\cos\theta - \cos\frac{\pi}{3}\sin\theta}$$

$$= \frac{\sqrt{3}/2}{\left(\sqrt{3}/2\right)\cos\theta - \left(1/2\right)\sin\theta}$$

$$= \frac{\sqrt{3}}{\sqrt{3}\cos\theta - \sin\theta}$$

bii When θ is a sufficiently small angle,

$$AD \approx \frac{\sqrt{3}}{\sqrt{3}\left(1-\frac{\theta^2}{2}\right)-\theta}$$

$$=\sqrt{3}\left(\sqrt{3}-\theta-\frac{\sqrt{3}}{2}\theta^2\right)^{-1}$$

$$=\left[1+\left(-\frac{\theta}{\sqrt{3}}-\frac{\theta^2}{2}\right)\right]^{-1}$$

$$\approx 1-\left(-\frac{\theta}{\sqrt{3}}-\frac{\theta^2}{2}\right)+\frac{(-1)(-2)}{2!}\left(-\frac{\theta}{\sqrt{3}}-\frac{\theta^2}{2}\right)^2$$

$$\approx 1+\frac{\theta}{\sqrt{3}}+\frac{\theta^2}{2}+\frac{\theta^2}{3}$$

$$=1+\frac{1}{\sqrt{3}}\theta+\frac{5}{6}\theta^2$$

$$\therefore a=\frac{1}{\sqrt{3}}, \qquad b=\frac{5}{6}$$

Q3

(i)

$$y=\ln(\cos ax-\sin ax)$$

$$e^{y}=\cos ax-\sin ax$$

$$e^{y}\frac{dy}{dx}=-a\sin ax-a\cos ax$$

$$e^{y}\frac{d^2y}{dx^2}+e^{y}\left(\frac{dy}{dx}\right)^2=-a^2\cos ax+a^2\sin ax$$

$$e^{y}\frac{d^2y}{dx^2}+e^{y}\left(\frac{dy}{dx}\right)^2=-a^2(\cos ax-\sin ax)$$

$$e^{y}\frac{d^2y}{dx^2}+e^{y}\left(\frac{dy}{dx}\right)^2=-a^2e^{y}$$

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+a^2=0$$

A majority of students produced very long, tedious and messy calculations by direct differentiation when the calculation could have been much simpler by rewriting the equation into the implicit form $e^{y}=\cos ax-\sin ax$ and applying implicit differentiation to obtain the desired equation. Students MUST learn SMART WAYS of doing math instead of using the brute force method

(ii)

$$\frac{d^3y}{dx^3}+2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}=0$$

When $x=0$, $y=0$

$$\frac{dy}{dx}=-a,\quad \frac{d^2y}{dx^2}=-2a^2,\quad \frac{d^3y}{dx^3}=-4a^3$$

$$y=-ax-a^2x^2-\frac{2}{3}a^3x^3+\cdots$$

Fairly straightforward application of Maclaurin's theorem to obtain series expansion.

(iii)

$$\ln(\cos 2x-\sin 2x)=-2x-4x^2-\frac{16}{3}x^3+\cdots$$

$$\cos 2x-\sin 2x\approx e^{-2x-4x^2}$$

$$\approx 1+(-2x-4x^2)+\frac{(-2x-4x^2)^2}{2!}\quad(\text{since } e^{x}\approx 1+x+\frac{x^2}{2!})$$

$$\approx 1-2x-4x^2+\frac{(-2x)^2}{2}$$

$$=1-2x-2x^2 \quad \text{where } k=-2$$

A number of students did not pay attention to the word 'Hence' which requires them to use an earlier result to deduce the next result. Many simply used the series expansions of sin x and cos x from MF26 which earn no credit

(iv)

$$\cos 2x-\sin 2x=1-\frac{(2x)^2}{2}-(2x)$$

$$=1-2x-2x^2$$

Some students forgot the '2' and wrote $\cos 2x-\sin 2x$ $=1-\frac{x^2}{2}-x$

Some used the double angle formulae for sin $2x$ and cos $2x$ which is not necessary