A Level H2 Math

Maclaurin Series Test 4

Q1

(a) By considering the Maclaurin expansion for $\cos x$, show that the expansion of $\sec x$ up to and including the term in x^4 is given by $1+\frac{1}{2}x^2+\frac{5}{24}x^4$. Hence show that the expansion for $\ln(\sec x)$ up to and including the term in x^4 is given by $\left[\frac{1}{2}x^2+Ax^4\right]$ where A is an unknown constant to be determined. [4]

(b) The variables x and y satisfy the conditions **(A)** and **(B)** as follows:

$$\left(1+x^2\right)\frac{\mathrm{d}y}{\mathrm{d}x}=1+y \quad \text{---}\textbf{(A)}$$

$$y=0 \text{ when } x=0 \quad \text{---}\textbf{(B)}$$

(i) Obtain the Maclaurin expansion of y, up to and including the term in x^3. [4]

(ii) Verify that both conditions **(A)** and **(B)** hold for the curve $\ln(1+y)=\tan^{-1}x$. [2]

(iii) Hence, without using a graphing calculator, find an approximation for $\int_0^{\frac{1}{2}}\left(e^{\tan^{-1}x}-1\right)\mathrm{d}x$. [2]

Q2

(i) The variables x and y are related by

$$(x+y)\frac{\mathrm{d}y}{\mathrm{d}x}+ky=2 \text{ and } y=1 \text{ at } x=0,$$

where k is a constant. Show that $(x+y)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+(1+k)\frac{\mathrm{d}y}{\mathrm{d}x}+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=0$. [1]

By further differentiation of this result, find the Maclaurin series for y, up to and including the term in x^3, giving the coefficients in terms of k. [4]

(ii) Given that x is small, find the series expansion of $\mathrm{g}(x)=\frac{1}{\sin^2\left(2x+\frac{\pi}{2}\right)}$ in ascending powers of x, up to and including the term in x^2.

If the coefficient of x^2 in the expansion of $\mathrm{g}(x)$ is equal to twice the coefficient of x^2 in the Maclaurin series for y found in part **(i)**, find the value of k. [4]

Q3

(i) Given that $y=\ln(1+\sin 2x)$, show that $\mathrm{e}^y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\mathrm{e}^y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2=-4\sin 2x$.

Find the first three non-zero terms in the Maclaurin's series for y. [5]

(ii) It is given that the three terms found in part **(i)** are equal to the first three terms in the series expansion of $ax(1+bx)^n$ for small x. Find the exact values of the constants a, b and n and use these values to find the coefficient of x^4 in the expansion of $ax(1+bx)^n$, giving your answer as a simplified rational number. [5]

Answers
Maclaurin Series Test 4

Q1

(a)

$$\sec x = \frac{1}{\cos x}$$

$$= \left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \ldots\right)^{-1}$$

$$= 1 + (-1)\left[-\frac{1}{2}x^2 + \frac{1}{24}x^4\right] + \frac{(-1)(-2)}{2!}\left[-\frac{1}{2}x^2 + \frac{1}{24}x^4\right]^2 + \ldots$$

$$= 1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 \quad + \frac{1}{4}x^4 + \ldots$$

$$= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 \text{ (up to } x^4\text{) (shown)}$$

$$\ln(\sec x) \approx \ln\left[1 + \frac{1}{2}x^2 + \frac{5}{24}x^4\right]$$

$$= \left[\frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots\right] - \frac{1}{2}\left[\frac{1}{2}x^2 + \frac{5}{24}x^4 + \ldots\right]^2$$

$$= \frac{1}{2}x^2 + \frac{5}{24}x^4 \quad - \frac{1}{2}\left(\frac{1}{4}x^4 \quad\right) + \ldots$$

$$= \frac{1}{2}x^2 + \frac{1}{12}x^4$$

Thus $A = \frac{1}{12}$

(b)(i) $\left(1+x^2\right)\frac{dy}{dx} = 1 + y$

$$\left(1+x^2\right)\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} = \frac{dy}{dx}$$

$$\left(1+x^2\right)\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} = (1-2x)\frac{d^2y}{dx^2} - 2\frac{dy}{dx}$$

At $x = 0, y = 0$

$$\frac{dy}{dx} = 1, \ \frac{d^2y}{dx^2} = 1, \ \frac{d^3y}{dx^3} = -1$$

Thus, $y = x + \frac{x^2}{2!} - \frac{x^3}{3!} + \ldots$

i.e. $y = x + \frac{x^2}{2} - \frac{x^3}{6} + \ldots$

(ii) $\ln(1+y) = \tan^{-1} x \Rightarrow \frac{1}{1+y}\frac{dy}{dx} = \frac{1}{1+x^2}$

$\therefore \left(1+x^2\right)\frac{dy}{dx} = 1 + y$ so condition **(A)** is satisfied.

At $x = 0$,

$\ln(1+y) = \tan^{-1} 0 = 0 \Rightarrow 1 + y = e^0$

$\therefore y = 0$

(iii) $\int_0^{\frac{1}{2}} \left(e^{\tan^{-1} x} - 1\right) dx \approx \int_0^{\frac{1}{2}} \left(x + \frac{x^2}{2} - \frac{x^3}{6}\right) dx = \frac{55}{384}$

Q2

(i)

$$(x+y)\frac{dy}{dx}+ky=2 \qquad \cdots(1)$$

Differentiating (1) w.r.t. x:

$$(x+y)\frac{d^2y}{dx^2}+\left(1+\frac{dy}{dx}\right)\frac{dy}{dx}+k\frac{dy}{dx}=0$$

$$(x+y)\frac{d^2y}{dx^2}+(1+k)\frac{dy}{dx}+\left(\frac{dy}{dx}\right)^2=0 \qquad \cdots(2)$$

Differentiating (2) w.r.t. x:

$$(x+y)\frac{d^3y}{dx^3}+\left(1+\frac{dy}{dx}\right)\frac{d^2y}{dx^2}+(1+k)\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right)=0$$

$$(x+y)\frac{d^3y}{dx^3}+\left(2+3\frac{dy}{dx}+k\right)\frac{d^2y}{dx^2}=0$$

$x=0,\quad y=1:\ \frac{dy}{dx}=2-k$

$$\frac{d^2y}{dx^2}=3k-6$$

$$\frac{d^3y}{dx^3}=6k^2-36k+48=6\left(k^2-6k+8\right)$$

$$\therefore y=1+(2-k)x+\left(\frac{3k-6}{2!}\right)x^2+\left(\frac{6\left(k^2-6k+8\right)}{3!}\right)x^3+\ldots$$

$$=1+(2-k)x+\left(\frac{3k-6}{2}\right)x^2+\left(k^2-6k+8\right)x^3+\ldots$$

(ii)

$$\sin\left(2x+\frac{\pi}{2}\right)=\sin 2x\cos\frac{\pi}{2}+\cos 2x\sin\frac{\pi}{2}=\cos 2x$$

$$\frac{1}{\sin^2\left(x+\frac{\pi}{2}\right)}=\frac{1}{\cos^2 2x}$$

$$\approx\left(1-\frac{(2x)^2}{2}\right)^{-2}$$

$$=\left(1-2x^2\right)^{-2}$$

$$=1+4x^2+\ldots$$

$$4=2\left(\frac{3k-6}{2}\right)$$

$$k=\frac{10}{3}$$

Q3

(i) $y=\ln(1+\sin 2x)$

so $e^y = 1+\sin 2x$

Differentiating with respect to x:

$$e^y \frac{dy}{dx} = 2\cos 2x$$
$$e^y \frac{d^2y}{dx^2} + e^y\left(\frac{dy}{dx}\right)^2 = -4\sin 2x \quad \text{(shown)}$$
$$e^y \frac{d^3y}{dx^3} + e^y\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + 2e^y\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + e^y\left(\frac{dy}{dx}\right)^3 = -8\cos 2x$$
$$e^y \frac{d^3y}{dx^3} + 3e^y\left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) + e^y\left(\frac{dy}{dx}\right)^3 = -8\cos 2x$$

When $x=0$, $e^y = 1 \Rightarrow y=0$, $\frac{dy}{dx}=2$, $\frac{d^2y}{dx^2}=-4$, $\frac{d^3y}{dx^3}=8$.

By Maclaurin's Theorem,

$$y = 2x - 4\left(\frac{x^2}{2!}\right) + 8\left(\frac{x^3}{3!}\right) + \cdots$$
$$= 2x - 2x^2 + \frac{4}{3}x^3 + \cdots$$

(ii)

$$ax(1+bx)^n$$
$$= ax\left[1 + n(bx) + \frac{n(n-1)}{2!}(bx)^2 + \frac{n(n-1)(n-2)}{3!}(bx)^3 + \ldots\right]$$
$$= ax + nabx^2 + \frac{n(n-1)}{2}ab^2x^3 + \ldots$$

By comparing coefficients, $a=2$

$$nab = -2 \Rightarrow nb = -1$$
$$\frac{n(n-1)}{2}ab^2 = \frac{4}{3}$$
$$\Rightarrow n^2b^2 - nb^2 = \frac{4}{3}$$
$$\Rightarrow (-1)^2 - (-1)b = \frac{4}{3}$$
$$\Rightarrow b = \frac{1}{3} \quad \text{and} \quad n = -3$$

x^4 term in the expansion of $2x\left(1+\frac{1}{3}x\right)^{-3} = 2x\left[\frac{-3(-4)(-5)}{3!}\left(\frac{1}{3}x\right)^3\right] = -\frac{20}{27}x^4$

$\therefore$ coefficient of $x^4 = -\frac{20}{27}$