A Level H2 Math

Maclaurin Series Test 3

Q1

(i) It is given that $\ln y = 2\sin x$. Show that $\frac{d^2y}{dx^2} = -y\ln y + \frac{1}{y}\left(\frac{dy}{dx}\right)^2$. [2]

(ii) Find the first four terms of the Maclaurin series for y in ascending powers of x. [4]

(iii) Using appropriate expansions from the List of Formulae (MF26), verify the expansion found in part **(ii)**. [2]

(iv) Given that x is sufficiently small for x^4 and higher powers of x to be neglected, deduce an approximation for $e^{(2\sin x)-\ln(\sec x)}$ in ascending powers of x. [2]

Q2

It is given that $y = \ln(1+\sin x)$.

(i) Find $\frac{dy}{dx}$. Show that $\frac{d^2y}{dx^2} = -e^{-y}$. [4]

(ii) Express $\frac{d^4y}{dx^4}$ in terms of $\frac{dy}{dx}$ and e^{-y}. [3]

(iii) Hence, find the first four non-zero terms in the Maclaurin series for $\ln(1+\sin x)$.[3]

Q3

A curve C has equation $y = f(x)$. The equation of the tangent to the curve C at the point where $x = 0$ is given by $2x - ay = 3$ where a is a positive constant.

It is also given that $y = f(x)$ satisfies the equation $(1+2x)\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$ and that the third term in the Maclaurin's expansion of $f(x)$ is $\frac{1}{3}x^2$.

Find the value of a. Hence, find the Maclaurin's series for $f(x)$ in ascending powers of x, up to and including the term in x^3. [7]

Answers
Maclaurin Series Test 3

Q1

(i)

Method 1

$\ln y = 2\sin x$

$\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos x$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 2y\cos x$

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -2y\sin x + 2\cos x\frac{\mathrm{d}y}{\mathrm{d}x} = -y\ln y + \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ (shown)

Method 2

$y = \mathrm{e}^{2\sin x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = (2\cos x)\mathrm{e}^{2\sin x}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = (2\cos x)y$

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -2y\sin x + 2\cos x\frac{\mathrm{d}y}{\mathrm{d}x}$

$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -y\ln y + \frac{1}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ (shown)

(ii)

$\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = -y\left(\frac{1}{y}\frac{\mathrm{d}y}{\mathrm{d}x}\right) - \ln y\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{1}{y^2}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 + \frac{2}{y}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)$

When $x = 0$, $y = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = 2$, $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 4$, $\frac{\mathrm{d}^3 y}{\mathrm{d}x^3} = 6$

$y = 1 + 2x + \frac{4x^2}{2!} + \frac{6x^3}{3!} + \ldots$

$y = 1 + 2x + 2x^2 + x^3 + \ldots$

(iii)

Method 1

$$y = e^{2\sin x}$$

$$= 1 + (2\sin x) + \frac{(2\sin x)^2}{2} + \frac{(2\sin x)^3}{6} + \dots$$

$$= 1 + 2(x - \frac{x^3}{6} + \dots) + \frac{[2(x+\dots)]^2}{2} + \frac{[2(x+\dots)]^3}{6} + \dots$$

$$= 1 + 2x - \frac{x^3}{3} + 2x^2 + \frac{4x^3}{3} + \dots$$

$$= 1 + 2x + 2x^2 + x^3 + \dots$$

Method 2

$$y = e^{2(x - \frac{x^3}{3!})}$$

$$= 1 + 2(x - \frac{x^3}{6}) + \frac{[2(x - \frac{x^3}{6})]^2}{2} + \frac{[2(x - \frac{x^3}{6})]^3}{6} + \dots$$

$$= 1 + 2x - \frac{2x^3}{6} + \frac{4x^2}{2} + \frac{8x^3}{6} + \dots$$

$$= 1 + 2x + 2x^2 + x^3 + \dots$$

(iv)

$$e^{(2\sin x) - \ln(\sec x)} = e^{(2\sin x)} e^{-\ln \sec x} = e^{(2\sin x)} e^{\ln \cos x}$$

$$= e^{(2\sin x)} \cos x$$

Method 1

$$e^{(2\sin x)} \cos x \approx (1 + 2x + 2x^2 + x^3)(1 - \tfrac{x^2}{2})$$

$$= 1 - \tfrac{x^2}{2} + 2x - \tfrac{2x^3}{2} + 2x^2 + x^3 + \dots$$

$$= 1 + 2x + \tfrac{3}{2}x^2 + \dots$$

Method 2

$$y = e^{2\sin x}$$

$$\frac{dy}{dx} = (2\cos x)e^{2\sin x}$$

$$\therefore \cos x e^{2\sin x} = \frac{1}{2}\frac{dy}{dx}$$

$$= \frac{1}{2}\frac{d}{dx}(1 + 2x + 2x^2 + x^3 + \dots)$$ ← From **(iii)**

$$= \frac{1}{2}(2 + 4x + 3x^2 + \dots)$$

$$= 1 + 2x + \tfrac{3}{2}x^2 + \dots$$

Q2

(i) $y = \ln(1+\sin x) \quad \Rightarrow e^y = 1+\sin x$

$$\frac{dy}{dx} = \frac{\cos x}{1+\sin x} \quad \text{[B1]}$$

$$\frac{d^2y}{dx^2} = \frac{(1+\sin x)(-\sin x)-(\cos x)(\cos x)}{(1+\sin x)^2}$$

$$= \frac{-\sin x-\sin^2 x-\cos^2 x}{(1+\sin x)^2}$$

$$= \frac{-\sin x-1}{(1+\sin x)^2} \quad \text{[A1]}$$

$$= \frac{-(\sin x+1)}{(1+\sin x)^2}$$

$$= \frac{-1}{1+\sin x}$$

$$= \frac{-1}{e^y}$$

$$= -e^{-y} \quad \text{(Shown)}$$

(ii)

$$\frac{d^3y}{dx^3} = -e^{-y}\left(-\frac{dy}{dx}\right)$$

$$= e^{-y}\frac{dy}{dx}$$

$$\frac{d^4y}{dx^4} = e^{-y}\frac{d^2y}{dx^2}+e^{-y}\left(-\frac{dy}{dx}\right)\left(\frac{dy}{dx}\right)$$

$$= e^{-y}(-e^{-y})-e^{-y}\left(\frac{dy}{dx}\right)^2 \quad \text{(from (i))}$$

$$= \underline{\underline{-(e^{-y})^2-e^{-y}\left(\frac{dy}{dx}\right)^2}} \text{ or } \underline{\underline{-e^{-y}\left[e^{-y}+\left(\frac{dy}{dx}\right)^2\right]}}$$

(iii) When $x=0$,

$$y = \ln 1 = 0$$

$$\frac{dy}{dx} = \frac{\cos 0}{1+\sin 0} = 1$$

$$\frac{d^2y}{dx^2} = -e^0 = -1$$

$$\frac{d^3y}{dx^3} = 1$$

$$\frac{d^4y}{dx^4} = -1-1 = -2$$

$$\therefore \ln(1+\sin x) = 0+x+\frac{(-1)}{2}x^2+\frac{1}{3!}x^3+\frac{(-2)}{4!}x^4+\ldots$$

$$= \underline{\underline{x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{12}x^4+\ldots}}$$

Q3

Curve C: $y = f(x)$

Tangent to C at $x = 0$ is $2x - ay = 3 \Rightarrow y = -\frac{3}{a} + \frac{2}{a}x$

Since the tangent to C at $x = 0$ is $y = f(0) + f'(0)x$,

$\therefore f(0) = -\frac{3}{a}$ and $f'(0) = \frac{2}{a}$

The 3rd term of the series for $f(x)$ is $\frac{1}{3}x^2$

$\Rightarrow \frac{f''(0)}{2!}x^2 = \frac{1}{3}x^2$

$\Rightarrow f''(0) = \frac{2}{3}$

From $(1+2x)\frac{d^2y}{dx^2} + y\frac{dy}{dx} = 0$,

When $x = 0$, we have $\frac{2}{3} + \left(-\frac{3}{a}\right)\left(\frac{2}{a}\right) = 0$

$\Rightarrow a^2 = 9$

$\Rightarrow a = 3$ (since $a > 0$)