A Level H2 Math

Maclaurin Series Test 2

Q1

(a) Given that the first two terms in the series expansion of $\sqrt{4-x}$ are equal to the first two terms in the series expansion of $p+\ln(q-x)$, find the constants p and q. [5]

(b)(i) Given that $y=\tan^{-1}(ax+1)$ where a is a constant, show that $\frac{\mathrm{d}y}{\mathrm{d}x}=a\cos^2 y$. Use this result to find the Maclaurin series for y in terms of a, up to and including the term in x^3. [5]

(ii) Hence, or otherwise, find the series expansion of $\frac{1}{1+(4x+1)^2}$ up to and including the term in x^2.

[3]

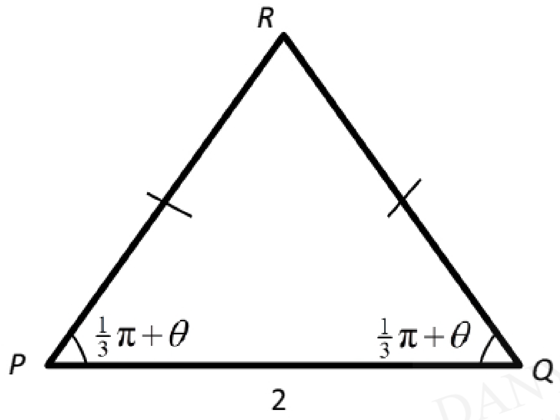
Q2

Using differentiation, find the Maclaurin's series of $\frac{e^{2x}}{1+x^2}$, in ascending powers of x up to and including x^3. [6]

Let $h(x)=\frac{e^{2x}}{1+x^2}$ and the cubic polynomial obtained above be $f(x)$.

Find, for $-2 \le x \le 2$, the set of values of x for which the value of $f(x)$ is within ± 0.5 of the value of $h(x)$. [3]

Q3



In the isosceles triangle PQR, $PQ = 2$ and the angle QPR = angle $PQR = \left(\frac{1}{3}\pi+\theta\right)$ radians. The area of triangle PQR is denoted by A.

Given that θ is a sufficiently small angle, show that

$$A = \frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta} \approx a + b\theta + c\theta^2,$$

for constants a, b and c to be determined in exact form. [5]

Answers
Maclaurin Series Test 2

Q1

(a) **Method ①:**

$$\sqrt{4-x}=4^{\frac{1}{2}}\left(1-\frac{x}{4}\right)^{\frac{1}{2}}$$
$$=2\left(1-\frac{x}{8}+\ldots\right)$$
$$=2-\frac{x}{4}+\ldots$$

$$p+\ln(q-x)=p+\ln\left[(q)\left(1-\frac{x}{q}\right)\right]$$
$$=p+\ln q+\ln\left(1-\frac{x}{q}\right)$$
$$=(p+\ln q)-\frac{x}{q}+\ldots$$

Comparing,

$2=p+\ln q$ and $-\frac{x}{4}=-\frac{x}{q}$

$2=p+\ln 4$ $\quad q=4$

$p=2-\ln 4$

Method ②:

Let $\mathrm{f}(x)=\sqrt{4-x}\Rightarrow \mathrm{f}'(x)=\frac{-1}{2\sqrt{4-x}}$, $\therefore \mathrm{f}(0)=2\ \&\ \mathrm{f}'(0)=-\frac{1}{4}$

Let $\mathrm{g}(x)=p+\ln(q-x)\Rightarrow \mathrm{g}'(x)=\frac{-1}{q-x}$, $\therefore \mathrm{g}(0)=p+\ln q\ \&\ \mathrm{f}'(0)=-\frac{1}{q}$

Comparing,

$q=4$ and $p=2-\ln 4$

Quite a majority of the students attempted this question successfully with a variety of methods. The most successful method being the use of repeated derivatives to form equations in p and q.

Common errors include erroneous use of the standard series expansions and also not knowing how to convert the expressions into the standard form required in their use.

A significant number of students also made arithmetic errors on the rules of logarithms, resulting in many marks lost.

(i)

$$y=\tan^{-1}(ax+1)$$
$$\tan y=ax+1$$
$$\sec^2 y\frac{\mathrm{d}y}{\mathrm{d}x}=a$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=a\cos^2 y \text{ (shown)}$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=2a\cos y(-\sin y)\frac{\mathrm{d}y}{\mathrm{d}x}=-a\sin 2y\frac{\mathrm{d}y}{\mathrm{d}x}$$
$$\frac{\mathrm{d}^3y}{\mathrm{d}x^2}=-2a\cos 2y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2-a\sin 2y\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$$

When $x=0$,

$$y=\tan^{-1}(1)=\frac{\pi}{4}$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=a\left(\cos\frac{\pi}{4}\right)^2=\frac{1}{2}a$$
$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=-a\left(\sin\frac{\pi}{2}\right)\left(\frac{1}{2}a\right)=-\frac{1}{2}a^2$$
$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3}=-2a\left(\cos\frac{\pi}{2}\right)\left(\frac{1}{2}a\right)^2-a\left(\sin\frac{\pi}{2}\right)\left(-\frac{1}{2}a^2\right)=\frac{1}{2}a^3$$

$$\tan^{-1}(ax+1)=\frac{\pi}{4}+\frac{1}{2}ax+\frac{-\frac{1}{2}a^2}{2!}x^2+\frac{\frac{1}{2}a^3}{3!}x^3+\ldots$$
$$=\frac{\pi}{4}+\frac{1}{2}ax-\frac{1}{4}a^2x^2+\frac{1}{12}a^3x^3+\ldots$$

Most students performed badly for this question as they are unclear about the process of implicit differentiation, often omitting the multiplication of the first derivative.

Students who attempted direct differentiation are rarely successful due to the complexity of the equations.

Most students who are successful with the repeated differentiation ended up with the correct expression, except a few who made arithmetic errors on the coefficients.

(ii) **Method ① (HENCE: direct differentiation using MF26)**

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\left(4x+1\right)\right]=\frac{4}{1+\left(4x+1\right)^2}$$

$$\frac{1}{1+\left(4x+1\right)^2}=\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan^{-1}\left(4x+1\right)\right]$$

$$=\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\pi}{4}+2x-4x^2+\frac{16}{3}x^3+\ldots\right]$$

$$=\frac{1}{4}\left[2-8x+16x^2+\ldots\right]$$

$$=\frac{1}{2}-2x+4x^2+\ldots$$

Method ② (OTHERWISE: binomial expansion)

$$\frac{1}{1+\left(4x+1\right)^2}=\left[1+\left(16x^2+8x+1\right)\right]^{-1}$$

$$=2^{-1}\left[1+\left(4x+8x^2\right)\right]^{-1}$$

$$=\frac{1}{2}\left[1-\left(4x+8x^2\right)+\frac{(-1)(-2)}{2!}\left(4x\right)^2+\ldots\right]$$

$$=\frac{1}{2}\left[1-4x-8x^2+16x^2+\ldots\right]$$

$$=\frac{1}{2}-2x+4x^2+\ldots$$

Method ③ (OTHERWISE: repeated differentiation)

$$\mathrm{f}(x)=\frac{1}{1+(4x+1)^2}\Rightarrow \mathrm{f}'(x)=\frac{-8(4x+1)}{\left[1+(4x+1)^2\right]^2}\Rightarrow \mathrm{f}(0)=\frac{1}{2}\ \&\ \mathrm{f}'(0)=-2$$

$$\Rightarrow \mathrm{f}''(x)=\frac{-32\left[1+(4x+1)^2\right]^2+128(4x+1)^2\left[1+(4x+1)^2\right]}{\left[1+(4x+1)^2\right]^4}\Rightarrow \mathrm{f}''(0)=8$$

$$\therefore \mathrm{f}(x)=\frac{1}{2}-2x+4x^2+\ldots$$

Most students were unable to see the link necessary for the "hence" method and adopted the otherwise methods. The most successful methods were those which involved repeated differentiation as it does not depend on the previous answers.

Many students attempted to use the series expansion for $(1+x)^n$ using $(4x+1)^2$ in place of x, but failing to realize that all powers of $(4x+1)$ will result in terms which have to be include (i.e. constant, x and x^2).

Q2

$$y=\frac{e^{2x}}{1+x^2}$$

$$\left(1+x^2\right)y=e^{2x}$$

$$\left(1+x^2\right)y'+2xy=2e^{2x}$$

$$\left(1+x^2\right)y''+2xy'+2xy'+2y=4e^{2x}$$

$$\Rightarrow\left(1+x^2\right)y''+4xy'+2y=4e^{2x}$$

$$\left(1+x^2\right)y'''+2xy''+4y'+4xy''+2y'=8e^{2x}$$

$$\Rightarrow\left(1+x^2\right)y'''+6xy''+6y'=8e^{2x}$$

When $x=0,\ y=1, y'=2, y''=2, y'''=-4$

$$\begin{aligned}y&=\frac{e^{2x}}{1+x^2}\\&=1+2x+2\left(\frac{x^2}{2!}\right)-4\left(\frac{x^3}{3!}\right)+\cdots\\&\approx 1+2x+x^2-\frac{2x^3}{3}\end{aligned}$$

$$a=2,\quad b=-\frac{2}{3}$$

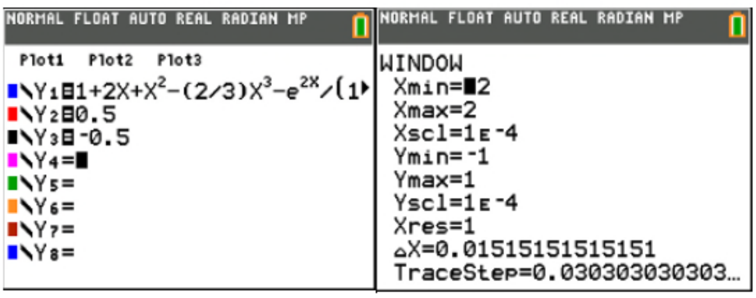
(a)

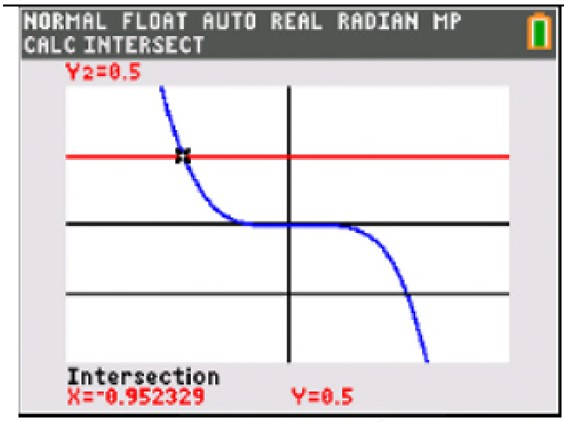
For $-2\le x\le 2$,

$$\left|f(x)-h(x)\right|\le 0.5$$

$$-0.5\le 1+2x+x^2-\frac{2x^3}{3}-\frac{e^{2x}}{1+x^2}\le 0.5$$

By GC,



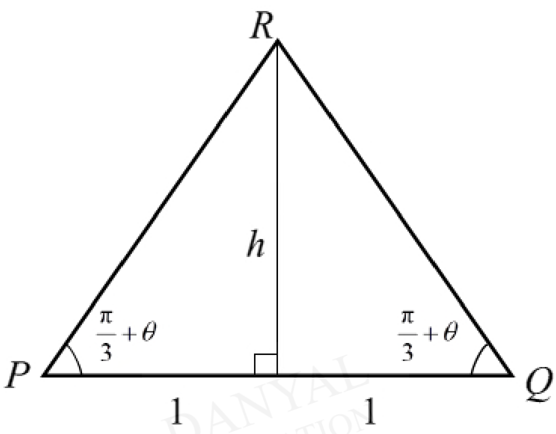


From the diagram above,

$$-0.95233 \leq x \leq 1.072619$$

$$\therefore -0.952 \leq x \leq 1.07$$

Q3



$$h=\tan\left(\frac{\pi}{3}+\theta\right)$$

$$A=\frac{1}{2}(2)\tan\left(\frac{\pi}{3}+\theta\right)=\tan\left(\frac{\pi}{3}+\theta\right)$$

$$=\frac{\tan\left(\frac{\pi}{3}\right)+\tan\theta}{1-\tan\left(\frac{\pi}{3}\right)\tan\theta}=\frac{\sqrt{3}+\tan\theta}{1-\sqrt{3}\tan\theta}\ \text{(shown)}$$

$$\approx\frac{\sqrt{3}+\theta}{1-\theta\sqrt{3}}$$

$$=\left(\sqrt{3}+\theta\right)\left(1-\theta\sqrt{3}\right)^{-1}$$

$$\approx\left(\sqrt{3}+\theta\right)\left(1+\theta\sqrt{3}+3\theta^2\right)$$

$$=\sqrt{3}+4\theta+\left(4\sqrt{3}\right)\theta^2$$