A Level H2 Math

Maclaurin Series Test 1

Q1

It is given that $e^y = (1+\sin x)^2$.

(i) Show that

$$e^y\left[\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]=2(\cos 2x-\sin x).$$

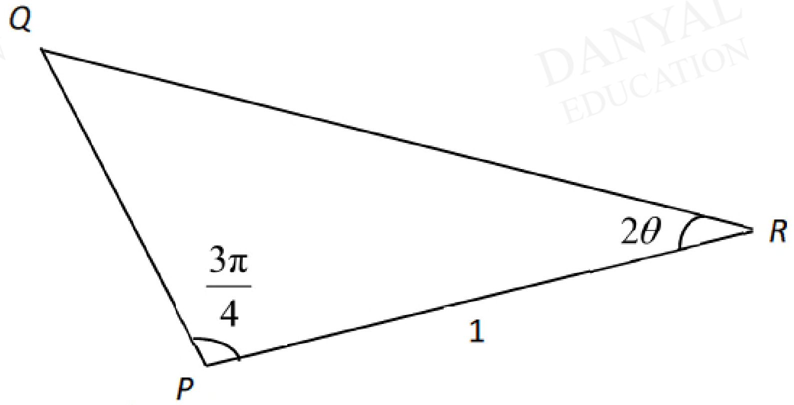
By repeated differentiation, find the series expansion of y in ascending powers of x, up to and including the term in x^3, simplifying your answer. [5]

(ii) Show how you can use the standard series expansion(s) to verify that the terms up to x^3 for your series expansion of y in **(i)** are correct. [3]

Q2

(a) Find the series expansion of $e^{2x}\ln(1+3x)$, where $-\frac{1}{3}<x\leq\frac{1}{3}$, in ascending powers of *x*, up to and including the term in x^3. [3]

(b) In the triangle *PQR* as shown in the diagram below, $PR=1$, angle $QPR=\frac{3\pi}{4}$ radians and angle $PRQ=2\theta$ radians.



(i) Show that $QR=\frac{1}{\cos 2\theta-\sin 2\theta}$. [4]

(ii) Given that θ is sufficiently small angle, show that $QR\approx 1+a\theta+b\theta^2$, for constants *a* and *b* to be determined. [4]

Q3

Let $y=f(x)$, where $f(x)=e^{\sqrt{(1-x)^3}}$ for $x\leq 1$.

Show that $4\sqrt{1-x}\frac{d^2y}{dx^2}+6(1-x)\frac{dy}{dx}-3y=0$. [4]

Hence find the Maclaurin series for f(x) up to and including the term in x^2. [3]

Using the standard series of e^x and $(1+x)^n$ given in the List of Formulae (MF26), show how you could verify the correctness of the series of $f(x)$ above. [4]

Answers
Maclaurin Series Test 1

Q1

(i)	$e^y = (1+\sin x)^2$ Differentiating w.r.t. x, $e^y \frac{dy}{dx} = 2(1+\sin x)\cos x$ $e^y \frac{dy}{dx} = 2\cos x + \sin 2x$ Differentiating w.r.t. x again, $e^y \frac{d^2y}{dx^2} + \frac{dy}{dx} e^y \frac{dy}{dx} = -2\sin x + 2\cos 2x$ $e^y \left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] = 2(\cos 2x - \sin x)$ (shown) Differentiating w.r.t. x: $e^y \left[\frac{d^3y}{dx^3} + 2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}\right] + \left[\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] e^y \frac{dy}{dx} = 2(-2\sin 2x - \cos x)$ Substituting $x = 0$, $y = 0;\quad \frac{dy}{dx} = 2;\quad \frac{d^2y}{dx^2} = -2;\quad \frac{d^3y}{dx^3} = 2$ $\Rightarrow y = 0 + 2x + \frac{-2}{2!}x^2 + \frac{2}{3!}x^3 + \ldots$ $\therefore y = 2x - x^2 + \frac{1}{3}x^3 + \ldots$	Most students can do the proof in the first part quite well although some have longer methods. Shorter method is to differentiate implicitly to get $e^y \frac{dy}{dx} = 2(1+\sin x)\cos x$ Most students could differentiate correctly $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2$ to get $\frac{d^3y}{dx^3} + 2\left(\frac{dy}{dx}\right)\frac{d^2y}{dx^2}$ A few fail to use the product rule to differentiate and got this part wrong.
(ii)	**Method 1**:	Common mistake made is to assume x is a small angle and use the small angle approximation. Correct approximation is

$$e^y = (1+\sin x)^2$$
$$\Rightarrow y = \ln(1+\sin x)^2$$
$$= 2\ln(1+\sin x)$$
$$= 2\ln\left(1+\left(x-\frac{x^3}{3!}\right)+\ldots\right)$$
$$= 2\left(\left(x-\frac{x^3}{3!}\right)-\frac{\left(x-\frac{x^3}{3!}\right)^2}{2}+\frac{\left(x-\frac{x^3}{3!}\right)^3}{3}+\ldots\right)$$
$$= 2\left(x-\frac{x^3}{6}-\frac{x^2}{2}+\frac{x^3}{3}+\ldots\right)$$
$$= 2x-x^2+\frac{1}{3}x^3+\ldots$$

which is same as the expansion for y found in (i), up to and including the term in $x^3 \Rightarrow$ verified.

Method 2:

$$\text{RHS} = (1+\sin x)^2$$
$$= \left(1+x-\frac{x^3}{3!}\right)^2$$
$$= 1+x-\frac{x^3}{6}+x+x^2-\frac{x^3}{6}+\ldots$$
$$= 1+2x+x^2-\frac{x^3}{3}+\ldots$$

$$\text{LHS} = e^y$$
$$= e^{\left(2x-x^2+\frac{1}{3}x^3+\ldots\right)}$$ (using expansion for y in (i))
$$= 1+\left(2x-x^2+\frac{1}{3}x^3\right)+\frac{\left(2x-x^2+\frac{1}{3}x^3\right)^2}{2!}+\frac{\left(2x-x^2+\frac{1}{3}x^3\right)^3}{3!}+\ldots$$
$$= 1+2x-x^2+\frac{1}{3}x^3+\frac{4x^2-2x^3-2x^3}{2}+\frac{8x^3}{6}+\ldots$$
$$= 1+2x+x^2-\frac{1}{3}x^3+\ldots$$

LHS = RHS $\Rightarrow$ verified.

$\sin x = x - \frac{x^3}{3!}$.

In some answers, detailed workings were not shown clearly.

Q2

(a)	$e^{2x}\ln(1+3x)$ $=\left(1+2x+\frac{(2x)^2}{2!}+\ldots\right)\left(3x-\frac{(3x)^2}{2}+\frac{(3x)^3}{3}-\ldots\right)$ where $-1<3x\le 1$ $=\left(1+2x+2x^2+\ldots\right)\left(3x-\frac{9}{2}x^2+9x^3-\ldots\right)$ $=3x-\frac{9}{2}x^2+9x^3+6x^2-9x^3+6x^3+\ldots$ $=3x+\frac{3}{2}x^2+6x^3+\ldots$ where $-\frac{1}{3}<x\le\frac{1}{3}$
(b)(i)	$\frac{QR}{\sin\frac{3\pi}{4}}=\frac{PR}{\sin\left(\pi-\frac{3\pi}{4}-2\theta\right)}$ $\frac{QR}{\sin\frac{3\pi}{4}}=\frac{PR}{\sin\left(\frac{\pi}{4}-2\theta\right)}$ $QR=\frac{\sin\frac{3\pi}{4}}{\sin\frac{\pi}{4}\cos 2\theta-\cos\frac{\pi}{4}\sin 2\theta}$ $QR=\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}\cos 2\theta-\frac{1}{\sqrt{2}}\sin 2\theta}$ $QR=\frac{1}{\cos 2\theta-\sin 2\theta}$ (shown)
(b)(ii)	When θ is small, $QR\approx\frac{1}{\left(1-\frac{(2\theta)^2}{2!}\right)-2\theta}$ $=\frac{1}{1-2\theta-2\theta^2}$ $=\left(1-\left(2\theta+2\theta^2\right)\right)^{-1}$ $=1+\left(2\theta+2\theta^2\right)+\left(2\theta+2\theta^2\right)^2+\ldots$ $=1+2\theta+2\theta^2+4\theta^2+\ldots$ $=1+2\theta+6\theta^2+\ldots$ $a=2,\ b=6$

Q3

$$y = e^{\sqrt{(1-x)^3}}$$

$$\frac{dy}{dx} = e^{\sqrt{(1-x)^3}}\left(\frac{3}{2}\right)(1-x)^{\frac{1}{2}}(-1) = \frac{-3}{2}y\sqrt{1-x}$$

$$\frac{d^2y}{dx^2} = \frac{-3}{2}\frac{dy}{dx}\sqrt{1-x} + \frac{-3}{2}y\frac{-1}{2\sqrt{1-x}} = \frac{3y}{4\sqrt{1-x}} - \frac{3\sqrt{1-x}}{2}\frac{dy}{dx}$$

$$4\sqrt{1-x}\frac{d^2y}{dx^2} = 3y - 6(1-x)\frac{dy}{dx}$$

Thus,

$$4\sqrt{1-x}\frac{d^2y}{dx^2} + 6(1-x)\frac{dy}{dx} - 3y = 0 \text{ (shown)}$$

When $x = 0$,

$$y = e$$

$$\frac{dy}{dx} = \frac{-3e}{2}$$

$$\frac{d^2y}{dx^2} = 3e$$

$$f(x) \approx f(0) + f'(0)x + \frac{f''(0)}{2!}x^2$$

$$\approx e - \frac{3e}{2}x + \frac{3e}{2}x^2$$

$$(1-x)^{\frac{3}{2}} \approx 1 - \frac{3}{2}x + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)}{2!}(-x)^2$$

$$= 1 - \frac{3}{2}x + \frac{3}{8}x^2$$

$$
\begin{aligned}
e^{\sqrt{(1-x)^3}} &\approx e^{1-\frac{3}{2}x+\frac{3}{8}x^2} \\
&= e\left(e^{-\frac{3}{2}x+\frac{3}{8}x^2}\right) \\
&\approx e\left(1+\left(-\frac{3}{2}x+\frac{3}{8}x^2\right)+\frac{\left(-\frac{3}{2}x+\frac{3}{8}x^2\right)^2}{2!}\right) \\
&\approx e\left(1-\frac{3}{2}x+\frac{3}{8}x^2+\frac{9}{8}x^2\right) \\
&= e\left(1-\frac{3}{2}x+\frac{3}{2}x^2\right) \\
&= e-\frac{3e}{2}x+\frac{3e}{2}x^2
\end{aligned}
$$

which is the same as the above series expansion of $f(x)$