A Level H2 Math

Integration Test 9

Q1

(a) Given that $\int_0^a x\sin x\,\mathrm{d}x = 0.5$, where $0 < a < 2$, find an equation that is satisfied by a and use it to find the value of a. [5]

(b) Write down a definite integral that represents the area of the region bounded by the curve with equation $y = \frac{\sqrt{x}}{3-\sqrt{x}}$, the two axes and the line $x = 4$.
Use the substitution $u = 3-\sqrt{x}$ to find the exact value of the area. [6]

Q2

A curve C has equation $y = \ln(x^2)$, $x \neq 0$.

(i) Sketch C. [2]

(ii) The part of C from the point $A(\mathrm{e}^{-1}, -2)$ to the point $B(\mathrm{e}^{\frac{k}{2}}, k)$, $k > 4$, and the line $y = -2$ is rotated about the y-axis to form the curved surface and the circular base of an open vase. Find the volume of the vase, giving your answer in terms of π and k, in exact form. [2]

(iii) Water flows into the vase at a constant rate of 2 cm^3 per second. By first showing that the volume of water in the vase is given by $V = \pi\left(x^2 - \mathrm{e}^{-2}\right)$ when the radius of the water surface is x cm, find the rate at which x is increasing, giving your answer in terms of x. [4]

(iv) An insect lands on the inner surface of the vase at the point (e, 2) just as the in-coming water reaches the depth of 2 cm. It immediately starts to crawl along C such that the x-coordinate of its location increases by a constant value of 0.03 cm per second. Find the coordinates of the point on C at which the insect will first come into contact with water. [5]

Q3

(i) By using the substitution $x-1 = 3\tan\theta$, find $\int \frac{1}{\sqrt{x^2-2x+10}}\,\mathrm{d}x$. [5]

(ii) By expressing $x+3 = A(2x-2)+B$, find $\int \frac{x+3}{\sqrt{x^2-2x+10}}\,\mathrm{d}x$. [3]

Integration Test 9

Answers

Q1

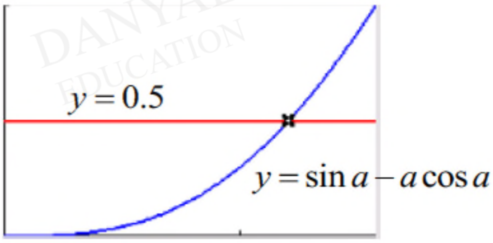
(a)

$$\int_0^a x\sin x\,dx = 0.5$$

$$\left[-x\cos x\right]_0^a + \int_0^a \cos x\,dx = 0.5$$

$$\left[-a\cos a + 0\right] + \left[\sin x\right]_0^a = 0.5$$

$$-a\cos a + \sin a = 0.5 \quad \text{--- (1)}$$



Using GC, a = 1.20249 =1.20 (3 s.f.)

(b)

$$\text{Area} = \int_0^4 \frac{\sqrt{x}}{3-\sqrt{x}}\,dx$$

Let $u = 3-\sqrt{x}$

$$\frac{du}{dx} = -\frac{1}{2\sqrt{x}} \Rightarrow \frac{dx}{du} = -2(3-u)$$

When $x = 0,\ u = 3$

When $x = 4,\ u = 1$

$$\int_0^4 \frac{\sqrt{x}}{3-\sqrt{x}}\,dx = \int_3^1 \left(\frac{3-u}{u}\right)\left[(-2)(3-u)\right]\,du$$

$$= \int_1^3 \frac{2(3-u)^2}{u}\,du$$

$$= 2\int_1^3 \frac{9-6u+u^2}{u}\,du$$

$$= 2\int_1^3 \left(\frac{9}{u} - 6 + u\right)du$$

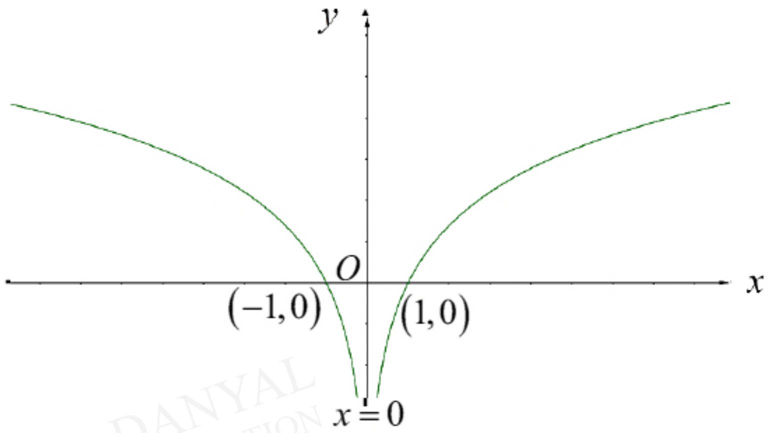
$$= 2\left[9\ln u - 6u + \frac{u^2}{2}\right]_1^3$$

$$= 2\left(9\ln 3 - 18 + \frac{9}{2}\right) - 2\left(-6 + \frac{1}{2}\right)$$

$$= 18\ln 3 - 16$$

Q2

i



ii

Volume of the vase $= \pi\int_{-2}^{k} x^2\,\mathrm{d}y$

$$= \pi\int_{-2}^{k} \mathrm{e}^y\,\mathrm{d}y$$

$$= \pi\left[\mathrm{e}^y\right]_{-2}^{k}$$

$$= \pi\left[\mathrm{e}^k - \mathrm{e}^{-2}\right]$$

iii

Volume of water, $V = \pi\int_{-2}^{y} \mathrm{e}^y\,\mathrm{d}y$

$$= \pi\left[\mathrm{e}^y - \mathrm{e}^{-2}\right]$$

$$= \pi\left[\mathrm{e}^{\ln x^2} - \mathrm{e}^{-2}\right]$$

$$= \pi\left[x^2 - \mathrm{e}^{-2}\right]$$

Given $\frac{\mathrm{d}V}{\mathrm{d}t} = 2$,

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \frac{\mathrm{d}V}{\mathrm{d}t} \times \frac{\mathrm{d}x}{\mathrm{d}V}$$

$$= 2 \times \frac{1}{2\pi x}$$

$$= \frac{1}{\pi x}$$

Hence the rate at which the radius of the water surface is increasing is $\frac{1}{\pi x}$ cm per second.

iv

For the insect, $\frac{dx}{dt} = 0.03$.

t seconds later, the location of the insect is at $x = 0.03t + e$

For the movement of the water,

$$\frac{dx}{dt} = \frac{1}{\pi x}$$

$$\int \pi x \; dx = \int 1 \; dt$$

$$\frac{\pi x^2}{2} = t + C$$

When $t = 0$, $x = 1$

$$C = \frac{\pi}{2}$$

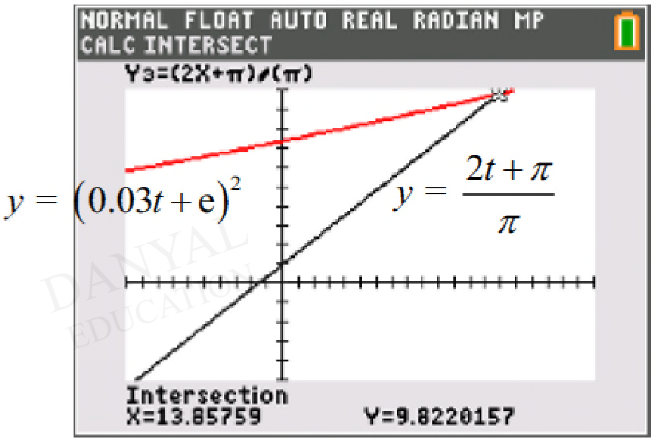
$$\frac{\pi x^2}{2} = t + \frac{\pi}{2}$$

When the insect first comes into contact with water,

$$\frac{\pi(0.03t + e)^2}{2} - \frac{\pi}{2} = t$$

$$\pi(0.03t + e)^2 - \pi = 2t$$

$$(0.03t + e)^2 = \frac{2t + \pi}{\pi}$$



Using GC, $t = 13.858$

$$x = 0.03(13.858) + e = 3.1340$$

$$y = \ln(3.1340)^2 = 2.28$$

Hence coordinates of the point = $(3.13, 2.28)$

Q3

4(i)

$x-1=3\tan\theta$

$\frac{\mathrm{d}x}{\mathrm{d}\theta}=3\sec^2\theta$

$\int \frac{1}{\sqrt{x^2-2x+10}}\,\mathrm{d}x=\int \frac{1}{\sqrt{(x-1)^2+3^2}}\,\mathrm{d}x$

Many candidates omitted the square root in the denominator when doing the substitution or applying the trigo identity.

$=\int \frac{1}{\sqrt{(3\tan\theta)^2+3^2}}\cdot 3\sec^2\theta\,\mathrm{d}\theta$

Apply
$\tan^2\theta+1=\sec^2\theta$

$=\int \frac{1}{3\sec\theta}\cdot 3\sec^2\theta\,\mathrm{d}\theta$

$=\int \sec\theta\,\mathrm{d}\theta$

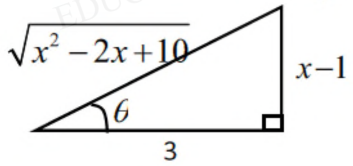
$=\ln\left|\sec\theta+\tan\theta\right|+C$

$=\ln\left|\frac{\sqrt{x^2-2x+10}}{3}+\frac{x-1}{3}\right|+C$

Remember the modulus sign, arbitrary constant and rewrite expression in terms of x.

Use the right angle triangle to give answer in terms of x.

$x-1=3\tan\theta \Rightarrow \tan\theta=\frac{x-1}{3}$



(ii)

$x+3=\frac{1}{2}(2x-2)+4$

$\int \frac{x+3}{\sqrt{x^2-2x+10}}\,\mathrm{d}x$

$=\int \frac{\frac{1}{2}(2x-2)+4}{\sqrt{x^2-2x+10}}\,\mathrm{d}x$

$=\frac{1}{2}\int \frac{2x-2}{\sqrt{x^2-2x+10}}\,\mathrm{d}x+\int \frac{4}{\sqrt{(x-1)^2+3^2}}\,\mathrm{d}x$

Standard form integral

$\int \underbrace{(2x-2)}_{f'(x)}\underbrace{(x^2-2x+10)}_{f(x)}^{-\frac{1}{2}}\mathrm{d}x=\frac{(x^2-2x+10)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1}+C$

$=\frac{1}{2}\frac{\sqrt{x^2-2x+10}}{\frac{1}{2}}+4\int \frac{1}{\sqrt{(x-1)^2+3^2}}\,\mathrm{d}x$

Many students erroneously applied the formula in MF26 when the form of the integral is not the same.

$=\sqrt{x^2-2x+10}+4\ln\left|\frac{\sqrt{x^2-2x+10}}{3}+\frac{x-1}{3}\right|+C$

Ans in (i)