A Level H2 Math

Integration Test 8

Q1
A curve C has parametric equations

$$x = \sin^2 t, \quad y = 2\cos t, \quad \text{for } 0 \le t \le \frac{\pi}{2}.$$

(i) Find a cartesian equation of C. [2]

The tangent to the curve at the point P where $t = \frac{\pi}{3}$ is denoted by l.

(ii) Find an equation of l. [3]

(iii) On the same diagram, sketch C and l, stating the coordinates of the axial intercepts and the point of intersection. [3]

The region R is bounded by the curve C, the line l and the y-axis.

(iv) Find the exact value of the volume of revolution formed when R is rotated completely about the x-axis. [3]

Q2

(a) **(i)** Find $\int e^x \cos nx \, dx$, where n is a positive integer. **[4]**

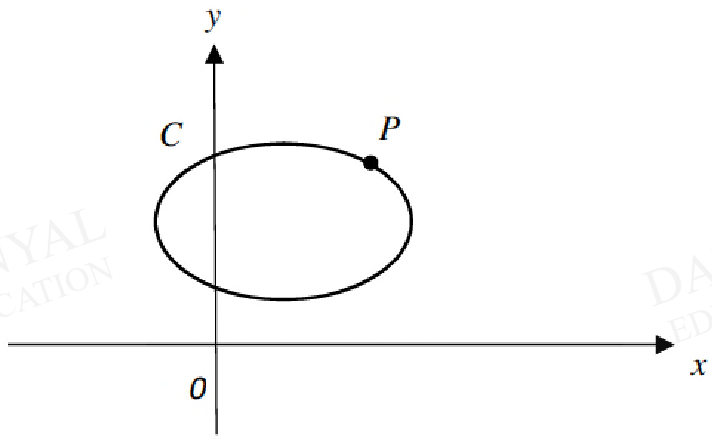
(ii) Hence, without the use of a calculator, find $\int_{\pi}^{2\pi} e^x \cos nx \, dx$ when n is odd. **[3]**

(b) The region bounded by the curve $y = \frac{\sqrt{x}}{16 - x^2}$, the y-axis and the line $y = \frac{\sqrt{2}}{12}$ is rotated 2π radians about the x-axis. Find the exact volume of the solid obtained. **[5]**

Q2

The diagram below shows the curve C with parametric equations

$$x = 1 + 2\sin\theta, \quad y = 4 + \sqrt{3}\cos\theta, \text{ for } -\pi < \theta \le \pi.$$



The point P is where $\theta = \dfrac{\pi}{6}$.

(i) Using a non-calculator method, find the equation of the normal at P. **[4]**

(ii) The normal at the point P cuts C again at point Q, where $\theta = \alpha$. Show that $8\sin\alpha - 2\sqrt{3}\cos\alpha = 1$ and hence deduce the coordinates of Q. **[3]**

(iii) Find the area of the region bounded by the curve C, the normal at point P and the vertical line passing through the point Q. **[4]**

Integration Test 8

Answers

Q1

(i) Using $\sin^2 t + \cos^2 t = 1$, a cartesian equation of C is

$$x + \left(\frac{y}{2}\right)^2 = 1$$

$$\Rightarrow y^2 = 4 - 4x,\ 0 \le x \le 1,\ 0 \le y \le 2$$

or

$$\Rightarrow y = 2\sqrt{1-x},\ 0 \le x \le 1$$

(ii) Differentiate with respect to x:

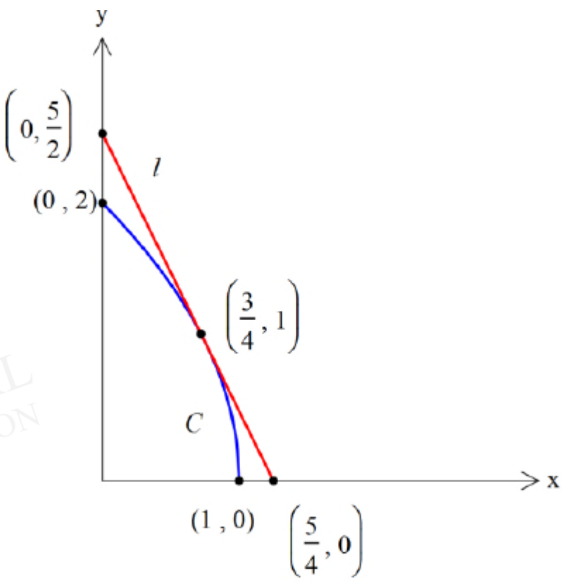
$$1 + \frac{y}{2}\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{2}{y}$$

When $t = \frac{\pi}{3}$, $x = \sin^2\left(\frac{\pi}{3}\right) = \frac{3}{4}$, $y = 2\cos\left(\frac{\pi}{3}\right) = 1$, $\frac{\mathrm{d}y}{\mathrm{d}x} = -2$

Hence, an equation of l is $y - 1 = -2\left(x - \frac{3}{4}\right)$

$$y = -2x + \frac{5}{2}$$

(iii)



(iv) Volume of revolution of R rotated about the x-axis

$$= \pi \int_0^{\frac{3}{4}} \left(-2x + \frac{5}{2}\right)^2 \, \mathrm{d}x - \pi \int_0^{\frac{3}{4}} (4 - 4x) \, \mathrm{d}x$$

$$= \pi \left[\frac{\left(-2x + \frac{5}{2}\right)^3}{3(-2)} \right]_0^{\frac{3}{4}} - \pi \left[4x - 2x^2 \right]_0^{\frac{3}{4}}$$

$$= -\frac{1}{6}\pi \left[1^3 - \left(\frac{5}{2}\right)^3 \right] - \pi \left[3 - 2\left(\frac{3}{4}\right)^2 \right]$$

$$= \frac{39}{16}\pi - \frac{15}{8}\pi$$

$$= \frac{9}{16}\pi \text{ units}^3$$

OR Use volume of cone $= \frac{1}{3}\pi r^2 h$, i.e.

$$\left[\frac{1}{3}\pi \left(\frac{5}{2}\right)^2 \left(\frac{5}{4}\right) - \frac{1}{3}\pi (1)^2 \left(\frac{1}{2}\right) \right] - \pi \int_0^{\frac{3}{4}} (4 - 4x) \, \mathrm{d}x$$

Q2

(a)

(i) Using integration by parts,

$\int e^x \cos nx \, dx$

$u = e^x$	$\dfrac{dv}{dx} = \cos nx$
$\dfrac{du}{dx} = e^x$	$v = \dfrac{\sin nx}{n}$

$= e^x\left(\dfrac{\sin nx}{n}\right) - \int \dfrac{e^x}{n}\sin nx \, dx$

$u = e^x$	$\dfrac{dv}{dx} = \sin nx$
$\dfrac{du}{dx} = e^x$	$v = -\dfrac{\cos nx}{n}$

$= e^x\left(\dfrac{\sin nx}{n}\right) - \dfrac{1}{n}\left[-\dfrac{e^x \cos nx}{n} + \int \dfrac{e^x \cos nx}{n} dx\right]$

$= e^x\left(\dfrac{\sin nx}{n}\right) + \dfrac{1}{n}\left(\dfrac{e^x \cos nx}{n}\right) - \dfrac{1}{n^2}\int e^x \cos nx \, dx$

Rearranging,

$\left(1+\dfrac{1}{n^2}\right)\int e^x \cos nx \, dx = e^x\left(\dfrac{\sin nx}{n}\right) + \dfrac{1}{n}\left(\dfrac{e^x \cos nx}{n}\right)$

$\int e^x \cos nx \, dx = \left(\dfrac{n^2}{1+n^2}\right)\left[e^x\left(\dfrac{\sin nx}{n}\right) + \dfrac{1}{n}\left(\dfrac{e^x \cos nx}{n}\right)\right] + c$ where c is a constant

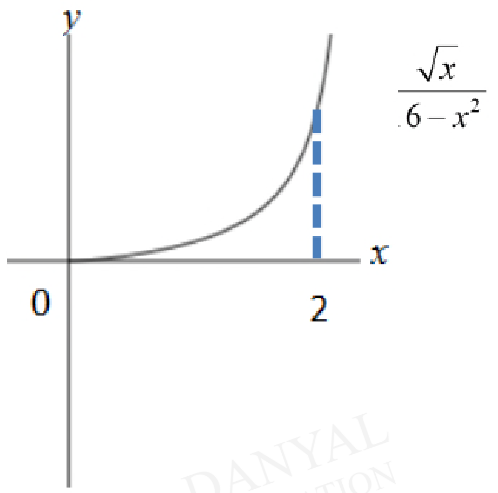
(ii) $\int_{\pi}^{2\pi} e^x \cos nx \, dx = \left(\dfrac{n^2}{1+n^2}\right)\left[e^x\left(\dfrac{\sin nx}{n}\right) + \dfrac{1}{n}\left(\dfrac{e^x \cos nx}{n}\right)\right]_{\pi}^{2\pi}$

$= \left(\dfrac{n^2}{1+n^2}\right)\left\{e^{2\pi}\left[\left(\dfrac{\sin 2n\pi}{n}\right) + \dfrac{1}{n}\left(\dfrac{\cos 2n\pi}{n}\right)\right] - e^{\pi}\left[\left(\dfrac{\sin n\pi}{n}\right) + \dfrac{1}{n}\left(\dfrac{\cos n\pi}{n}\right)\right]\right\}$

For any positive integer n, $\sin 2n\pi = 0$ and $\cos 2n\pi = 1$
If n is odd, $\sin n\pi = 0$ and $\cos n\pi = -1$

$\int_{\pi}^{2\pi} e^x \cos nx \, dx = \left(\dfrac{n^2}{1+n^2}\right)\left[e^{2\pi}\left(0+\dfrac{1}{n^2}\right) - e^{\pi}\left(0-\dfrac{1}{n^2}\right)\right] = \left(\dfrac{1}{1+n^2}\right)\left(e^{2\pi}+e^{\pi}\right)$ (Ans)

(b)



$$y=\frac{\sqrt{x}}{16-x^2} \Rightarrow y^2=\frac{x}{\left(16-x^2\right)^2}$$

Hence volume required

$$=\pi r^2 h-\pi\int_0^2 y^2\,\mathrm{d}x$$

$$=\pi\left(\frac{\sqrt{2}}{12}\right)^2(2)-\pi\int_0^2\frac{x}{\left(16-x^2\right)^2}\,\mathrm{d}x$$

$$=\pi\left(\frac{\sqrt{2}}{12}\right)^2(2)-\frac{\pi}{(-2)}\int_0^2\frac{-2x}{\left(16-x^2\right)^2}\,\mathrm{d}x$$

$$=\pi\left(\frac{\sqrt{2}}{12}\right)^2(2)+\frac{\pi}{2}\left[\frac{\left(16-x^2\right)^{-1}}{-1}\right]_0^2$$

$$=\frac{4}{144}\pi+\frac{\pi}{2}\left[-\frac{1}{12}+\frac{1}{16}\right]$$

$$=\frac{5\pi}{288}\text{ units}^3$$

Q3

(i)

$$\frac{dx}{d\theta} = 2\cos\theta \ , \quad \frac{dy}{d\theta} = -\sqrt{3}\sin\theta$$

$$\frac{dy}{dx} = \frac{-\sqrt{3}\sin\theta}{2\cos\theta} = -\frac{\sqrt{3}}{2}\tan\theta$$

When $\theta = \frac{\pi}{6}$, $x = 2$, $y = \frac{11}{2}$, $\frac{dy}{dx} = -\frac{1}{2}$

Equation of normal : $y - \left(\frac{11}{2}\right) = 2(x-2)$

$$y = 2x + \frac{3}{2}$$

(ii)

$x = 1 + 2\sin\theta \cdots\cdots(1)$

$y = 4 + \sqrt{3}\cos\theta \cdots\cdots(2)$

Substitute equation (1) and (2) into $y = 2x + \frac{3}{2}$

$$4 + \sqrt{3}\cos\theta = 2(1 + 2\sin\theta) + \frac{3}{2}$$

$$\frac{1}{2} + \sqrt{3}\cos\theta = 4\sin\theta$$

$$8\sin\theta - 2\sqrt{3}\cos\theta = 1$$

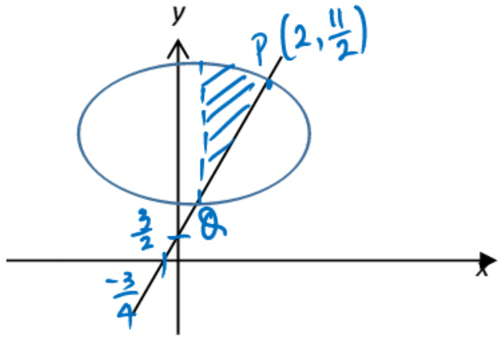
At Point Q, $\theta = \alpha$

$8\sin\alpha - 2\sqrt{3}\cos\alpha = 1$ (shown)

Using GC: $\alpha = -2.847916$ or $\alpha = 0.52359$ (Reject, same as $\frac{\pi}{6}$, point P)

Hence, using GC
coordinates of Q (0.42105, 2.3421)
Q (0.421, 2.34)

(iii)



when $x = 0.42105$

$0.42105 = 1 + 2\sin\theta$

$\sin\theta = -0.289475$

$\theta = -0.29368$ or -2.8479 (at point Q)

$$= \int_{0.42105}^{2} y_1 \, \mathrm{d}x - \int_{0.42105}^{2} y_2 \, \mathrm{d}x$$

Required Area $= \int_{-0.29368}^{\frac{\pi}{6}} \left(4 + \sqrt{3}\cos\theta\right) \left(2\cos\theta\right) \mathrm{d}\theta - \int_{0.42105}^{2} \left(2x + \frac{3}{2}\right) \mathrm{d}x$

$= 8.9613 - 6.1911$

$= 2.7702 \approx 2.77$ units^2 (3 s.f.)