A Level H2 Math

Integration Test 7

Q1

(i) Find $\int n\cos^{-1}(nx)\,\mathrm{d}x$, where n is a positive constant. [3]

(ii) Hence find the exact value of $\int_0^{\frac{1}{2n}} n\cos^{-1}(nx)\,\mathrm{d}x$. [2]

Q2

Given that $\mathrm{f}(x)=\sin 2x+\cos 2x$, express $\mathrm{f}(x)$ as $R\sin(2x+\alpha)$, where $R>0$, $0<\alpha<\frac{\pi}{2}$ and R and α are constants to be found. [2]

(i) Describe a sequence of transformations involved that transformed $y=\sin x$ to $y=\mathrm{f}(x)$. [3]

(ii) Sketch the graph of $y=\mathrm{f}(x)$ for $0\le x\le\frac{3\pi}{8}$, indicating clearly the exact coordinates of the maximum point and the end points of the graph. [3]

(iii) The region bounded by the curve $y=\mathrm{f}(x)$, the line $x=\frac{\pi}{8}$ and both axes is rotated about the y-axis through 2π radians. Find the volume of the solid of revolution correct to 4 decimal places. [4]

Q3

(a) Find $\int \frac{x+2}{\sqrt{(1-8x-4x^2)}}\,\mathrm{d}x$. [4]

(b) Use the substitution $x=2\sec\theta$ to find the exact value of $\int_2^4 \frac{1}{x}\sqrt{(x^2-4)}\,\mathrm{d}x$. [4]

Integration Test 7

Answers

Q1

(i)

$$\int n\cos^{-1}(nx)\,dx$$

$$=(nx)\cos^{-1}(nx)-\int(nx)\left(-\frac{n}{\sqrt{1-(nx)^2}}\right)dx$$

$$=(nx)\cos^{-1}(nx)-\frac{1}{2}\int(-2n^2x)(1-n^2x^2)^{-1/2}\,dx$$

$$=(nx)\cos^{-1}(nx)-\frac{1}{2}\times\frac{(1-n^2x^2)^{1/2}}{\frac{1}{2}}+C$$

$$=(nx)\cos^{-1}(nx)-\sqrt{(1-n^2x^2)}+C$$

(ii)

$$\int_0^{\frac{1}{2n}} n\cos^{-1}(nx)\,dx$$

$$=\left[(nx)\cos^{-1}(nx)-\sqrt{(1-n^2x^2)}\right]_0^{\frac{1}{2n}}$$

$$=\left[\frac{1}{2}\cos^{-1}\frac{1}{2}-\sqrt{1-\frac{1}{4}}\right]-(0-1)$$

$$=\frac{\pi}{6}-\frac{\sqrt{3}}{2}+1 \quad or \quad \frac{\pi}{6}+\frac{2-\sqrt{3}}{2}$$

Q2

$f(x) = \sin 2x + \cos 2x$

$R = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\tan\alpha = 1 \quad \Rightarrow \quad \alpha = \dfrac{\pi}{4}$

$f(x) = \sin 2x + \cos 2x = \sqrt{2}\sin\left(2x + \dfrac{\pi}{4}\right)$

(i)

Transforming $y = \sin x$ to $y = \sqrt{2}\sin\left(2x + \dfrac{\pi}{4}\right)$

Sequence of Transformation:

Either

A: A translation of $\dfrac{\pi}{4}$ units in the negative x-direction

B: A scaling/stretch with scale factor $\dfrac{1}{2}$ parallel to the x-axis.

C: A scaling/stretch with scale factor $\sqrt{2}$ parallel to the y-axis.

Acceptable sequence: ABC, ACB, CAB.

OR $y = \sqrt{2}\sin\left[2\left(x + \dfrac{\pi}{8}\right)\right]$

D: A scaling/stretch with scale factor $\dfrac{1}{2}$ parallel to the x-axis.

E: A translation of $\dfrac{\pi}{8}$ units in the negative x-direction.

F: A scaling/stretch with scale factor $\sqrt{2}$ parallel to the y-axis.

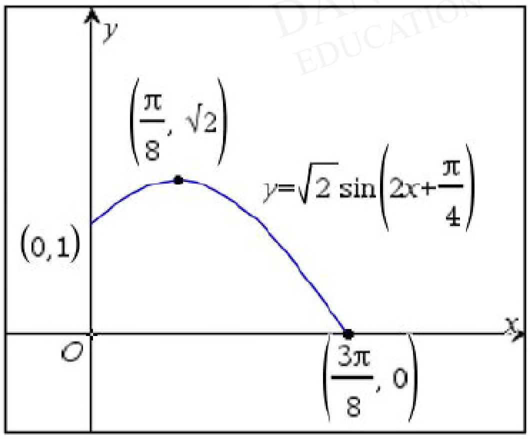
Acceptable sequence: DEF, DFE, FDE

(ii)

Max point occurs when $\sin\left(2x + \dfrac{\pi}{4}\right) = 1$

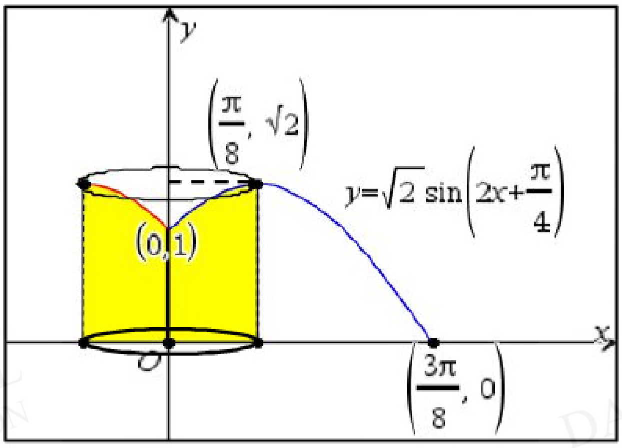
$\Rightarrow \quad \left(2x + \dfrac{\pi}{4}\right) = \dfrac{\pi}{2}$

$\Rightarrow \quad x = \dfrac{\pi}{8}, y = \sqrt{2}$



(iii)

$y=\sqrt{2}\sin\left(2x+\frac{\pi}{4}\right)$



The curve is one-one for $0 \le x \le \frac{\pi}{8}$,

thus inverse function exists.

$\sin\left(2x+\frac{\pi}{4}\right)=\frac{y}{\sqrt{2}}$

$2x+\frac{\pi}{4}=\sin^{-1}\frac{y}{\sqrt{2}}$

$x=\frac{1}{2}\left[\sin^{-1}\left(\frac{y}{\sqrt{2}}\right)-\frac{\pi}{4}\right]$

Volume = Volume of cylinder - $\pi\int_1^{\sqrt{2}} x^2 \ \mathrm{d}y$

$=\pi\left(\frac{\pi}{8}\right)^2\sqrt{2}-\pi\int_1^{\sqrt{2}}\frac{1}{4}\left[\sin^{-1}\left(\frac{y}{\sqrt{2}}\right)-\frac{\pi}{4}\right]^2 \ \mathrm{d}y$

$=0.6506458$

$\approx 0.6506 \ (4 \ \text{d.p.})$

Q3

(a) $\int \frac{x+2}{\sqrt{1-8x-4x^2}}\,dx$

$= -\frac{1}{8}\int \frac{-8x-8}{\sqrt{1-8x-4x^2}}\,dx + \int \frac{1}{\sqrt{1-8x-4x^2}}\,dx$

$= -\frac{1}{8}\int \frac{-8x-8}{\sqrt{1-8x-4x^2}}\,dx + \int \frac{1}{\sqrt{5-4(x+1)^2}}\,dx$

$= -\frac{1}{4}\sqrt{1-8x-4x^2} + \frac{1}{2}\sin^{-1}\frac{2\sqrt{5}(x+1)}{5} + C$

(b) $x = 2\sec\theta \Rightarrow \frac{dx}{d\theta} = 2\sec\theta\tan\theta$

When $x = 2$, $\sec\theta = 1 \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$

When $x = 4$, $\sec\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$

$\int_2^4 \frac{1}{x}\sqrt{(x^2-4)}\,dx$

$= \int_0^{\frac{\pi}{3}} \frac{\sqrt{4\sec^2\theta - 4}}{2\sec\theta}(2\sec\theta\tan\theta)\,d\theta$

$= \int_0^{\frac{\pi}{3}} \frac{2\tan\theta}{2\sec\theta}(2\sec\theta\tan\theta)\,d\theta$

since $\sqrt{4\sec^2\theta-4} = 2\sqrt{\tan^2\theta} = 2\tan\theta$ for $0 \le \theta \le \frac{\pi}{3}$ $= \int_0^{\frac{\pi}{3}} 2\tan^2\theta\,d\theta$

$= 2\int_0^{\frac{\pi}{3}} \sec^2\theta - 1\,d\theta$

$= 2[\tan\theta - \theta]_0^{\pi/3}$

$= 2\left[\sqrt{3} - \frac{\pi}{3}\right]$