A Level H2 Math

Integration Test 6

Q1

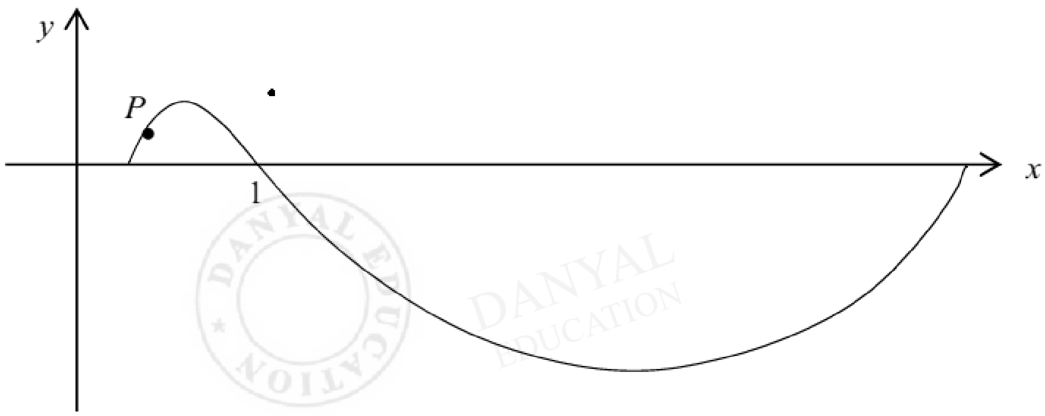
(i) Show by integration that

$$\int e^{-2x}\sin x \ dx = -\frac{2}{5}e^{-2x}\sin x - \frac{1}{5}e^{-2x}\cos x + A$$

where A is an arbitrary constant. [3]

The diagram below shows a sketch of curve C, with parametric equations

$$x = e^{-t}\ ,\ y = e^{-t}\sin t\ ,\ -\pi \le t \le \pi .$$



Point P lies on C where $t = \frac{\pi}{2}$.

(ii) Find the equation of the normal at P. [3]

(iii) Find the exact area bounded by the curve C for $0 \le t \le \pi$, the line $x = 1$ and the normal at P. [5]

(iv) The normal at P cuts the curve C again at two points where $t = q$ and $t = r$. Find the values of q and r. [3]

Q2

(a) Using the substitution $u = 2x+3$, find $\int \frac{x}{(2x+3)^3}\,dx$ in the for $-\frac{Px+Q}{R(2x+3)^2}+c$ where P, Q and R are positive integers to be determined. [3]

Hence find $\int \frac{x\ln(4x+3)}{(2x+3)^3}\,dx$. [3]

(b) Find $\int \sin 4x \cos 6x\,dx$. [2]

Hence or otherwise, find $\int e^x \sin 4e^x \cos 6e^x\,dx$. [1]

Q3

(i) Find $\int_2^n \frac{9x}{(x^2-1)^3}\,dx$, where $n \geq 2$ and hence evaluate $\int_2^\infty \frac{9x}{(x^2-1)^3}\,dx$. [3]

(ii) Sketch the curve $y=\frac{9x}{(x^2-1)^3}$ for $x \geq 0$. [2]

(iii) The region R is bounded by the curve, the line $y=\frac{2}{3}$ and the line $x=5$.

Write down the equation of the curve when it is translated by $\frac{2}{3}$ units in the negative y-direction. [1]

Hence or otherwise, find the volume of the solid formed when R is rotated completely about the line $y=\frac{2}{3}$, leaving your answer correct to 3 decimal places. [2]

Integration Test 6

Answers

Q1

i

$$\int e^{-2x}\sin x \ \mathrm{d}x$$
$$=(-\cos x)(e^{-2x})-\int(-\cos x)(-2e^{-2x})\ \mathrm{d}x$$
$$=-e^{-2x}\cos x-2\left[(\sin x)(e^{-2x})-\int\sin x(-2e^{-2x})\ \mathrm{d}x\right]$$
$$=-e^{-2x}\cos x-2e^{-2x}\sin x-4\int e^{-2x}\sin x\ \mathrm{d}x$$
$$5\int e^{-2x}\sin x\ \mathrm{d}x=-e^{-2x}\cos x-2e^{-2x}\sin x+C$$
$$\int e^{-2x}\sin x\ \mathrm{d}x=-\frac{2}{5}e^{-2x}\sin x-\frac{1}{5}e^{-2x}\cos x+A$$

ii

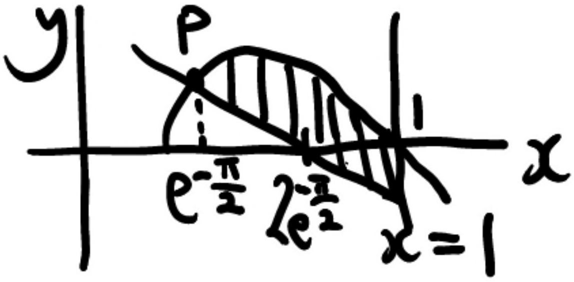
$$\frac{\mathrm{d}x}{\mathrm{d}t}=-e^{-t} \qquad \frac{\mathrm{d}y}{\mathrm{d}t}=-e^{-t}\sin t+e^{-t}\cos t$$
$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{-e^{-t}\sin t+e^{-t}\cos t}{-e^{-t}}=\sin t-\cos t$$

At $t=\frac{\pi}{2}$, $\frac{\mathrm{d}y}{\mathrm{d}x}=\sin\left(\frac{\pi}{2}\right)-\cos\left(\frac{\pi}{2}\right)=1-0=1$, so gradient of normal = -1

$x=e^{-\pi/2}$, $\qquad y=e^{-\pi/2}\sin\frac{\pi}{2}=e^{-\pi/2}$

Equation of normal: $\quad y-e^{-\pi/2}=-1(x-e^{-\pi/2}) \Rightarrow y=-x+2e^{-\pi/2}$

iii



Area

$$= \int_{e^{-\pi/2}}^{1} e^{-t}\sin t - \left(-x + 2e^{-\pi/2}\right) dx$$

$$= \int_{\pi/2}^{0} e^{-t}\sin t(-e^{-t})dt + \int_{e^{-\pi/2}}^{1} x - 2e^{-\pi/2}\ dx$$

$$= -\int_{\pi/2}^{0} e^{-2t}\sin t\ dt + \left[\frac{x^2}{2}\right]_{e^{-\pi/2}}^{1} - \left[2e^{-\pi/2}x\right]_{e^{-\pi/2}}^{1}$$

$$= -\left[-\frac{2}{5}e^{-2x}\sin x - \frac{1}{5}e^{-2x}\cos x\right]_{\pi/2}^{0} + \left[\frac{1}{2} - \frac{e^{-\pi}}{2}\right] - \left[2e^{-\pi/2}\left(1 - e^{-\pi/2}\right)\right]$$

$$= \frac{2}{5}e^{0}\sin 0 + \frac{1}{5}e^{0}\cos 0 - \frac{2}{5}e^{-\pi}\sin\left(\frac{\pi}{2}\right) - \frac{1}{5}e^{-\pi}\cos\frac{\pi}{2} + \frac{1}{2} - \frac{e^{-\pi}}{2} - 2e^{-\pi/2} + 2e^{-\pi}$$

$$= \frac{1}{5} - \frac{2}{5}e^{-\pi} + \frac{1}{2} - \frac{e^{-\pi}}{2} - 2e^{-\pi/2} + 2e^{-\pi}$$

$$= \frac{11}{10}e^{-\pi} - 2e^{-\pi/2} + \frac{7}{10}$$

Alternative:

$$\text{Area} = \int_{e^{-\pi/2}}^{1} e^{-t}\sin t - \frac{1}{2}(e^{-\pi/2})(2e^{-\pi/2} - e^{-\pi/2}) + \frac{1}{2}(1 - 2e^{-\pi/2})^2$$

[When x =1, y = 1+ $2e^{-\pi/2}$]

iv

For normal to meet curve again,

Substitute parametric eqns into $y = -x + 2e^{-\pi/2}$

$$e^{-t}\sin t = -e^{-t} + 2e^{-\pi/2}$$

$$e^{-t}(\sin t + 1) - 2e^{-\pi/2} = 0$$

Using GC, $t = -1.92148,\ \ -1.0145,\ \ 1.5707$ (rej, this is $\frac{\pi}{2}$)

So $q = -1.92$ and $r = -1.01$ (to 3 sf)

Q2

(a) Given $u = 2x + 3 \Rightarrow \frac{du}{dx} = 2$

$$\int \frac{x}{(2x+3)^3}\, dx = \int \frac{\frac{1}{2}(u-3)}{u^3} \cdot \frac{1}{2}\, du$$

$$= \frac{1}{4}\int \left[u^{-2} - 3u^{-3}\right] du$$

$$= \frac{1}{4}\left[-u^{-1} + \frac{3}{2}u^{-2}\right] + C$$

$$= -\frac{1}{4(2x+3)} + \frac{3}{8(2x+3)^2} + C$$

$$= \frac{-2(2x+3)+3}{8(2x+3)^2} + C$$

$$= -\frac{4x+3}{8(2x+3)^2} + C$$

$P = 4$, $Q = 3$ and $R = 8$

$$\int \frac{\ln(4x+3)^x}{(2x+3)^3}\ \mathrm{d}x$$

$$= \int \frac{x}{(2x+3)^3} \cdot \ln(4x+3)\ \mathrm{d}x \qquad \text{Let } \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{x}{(2x+3)^3}, u = \ln(4x+3)$$

$$= -\frac{(4x+3)\ln(4x+3)}{8(2x+3)^2} - \int -\frac{(4x+3)}{8(2x+3)^2} \cdot \frac{4}{(4x+3)}\mathrm{d}x\ +\ C$$

$$= -\frac{(4x+3)\ln(4x+3)}{8(2x+3)^2} + \frac{1}{2}\int (2x+3)^{-2}\ \mathrm{d}x\ +\ C$$

$$= -\frac{(4x+3)\ln(4x+3)}{8(2x+3)^2} + \frac{1}{2}(2x+3)^{-1}\left(-\frac{1}{2}\right) + C$$

$$= -\frac{(4x+3)\ln(4x+3)}{8(2x+3)^2} - \frac{1}{4(2x+3)} + C$$

$$= -\frac{(4x+3)\ln(4x+3) + 2(2x+3)}{8(2x+3)^2} + C$$

(b) $\int \sin 4x \cos 6x\ \mathrm{d}x$

$$= \frac{1}{2}\int \sin 10x + \sin(-2x)\ \mathrm{d}x$$

$$= \frac{1}{2}\int \sin 10x - \sin 2x\ \mathrm{d}x$$

$$= \frac{1}{2}\left[-\frac{1}{10}\cos 10x + \frac{1}{2}\cos 2x\right] + C$$

$$= -\frac{1}{20}\cos 10x + \frac{1}{4}\cos 2x + C$$

$$\int e^x \sin 4e^x \cos 6e^x\ \mathrm{d}x$$

$$= -\frac{1}{20}\cos 10e^x + \frac{1}{4}\cos 2e^x + C$$

Q3

(i)

$$\int_2^n \frac{9x}{(x^2-1)^3}\ \mathrm{d}x = \frac{9}{2}\int_2^n \frac{2x}{(x^2-1)^3}\ \mathrm{d}x$$

$$= \frac{9}{2}\left[-\frac{1}{2}(x^2-1)^{-2}\right]_2^n$$

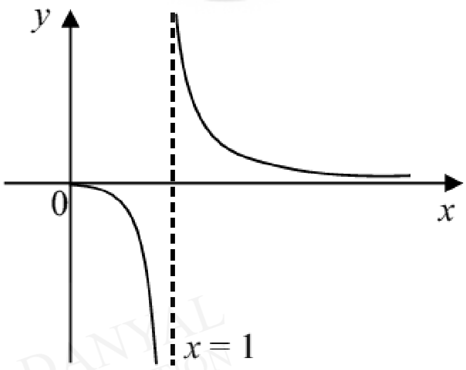
$$= \frac{9}{2}\left[-\frac{1}{2(n^2-1)^2}+\frac{1}{18}\right]$$

$$= \frac{1}{4}-\frac{9}{4(n^2-1)^2}$$

$$\lim_{n\to\infty}\left[\int_2^n \frac{9x}{(x^2-1)^3}\ \mathrm{d}x\right] = \lim_{n\to\infty}\left[\frac{1}{4}-\frac{9}{4(n^2-1)^2}\right]$$

$$= \frac{1}{4}$$

(ii)



(iii) The equation of the transformed curve is $y=\frac{9x}{(x^2-1)^3}-\frac{2}{3}$.

Volume of revolution $=\pi\int_2^5\left(\frac{9x}{((x^2-1)^3}-\frac{2}{3}\right)^2\ \mathrm{d}x\ = 3.385\text{ units}^3$ (to 3 d.p.)