A Level H2 Math

Integration Test 5

Q1

(a) A curve is defined parametrically by the equations

$$x = \sin t \quad \text{and} \quad y = \cos^3 t, \quad -\pi \le t \le \pi.$$

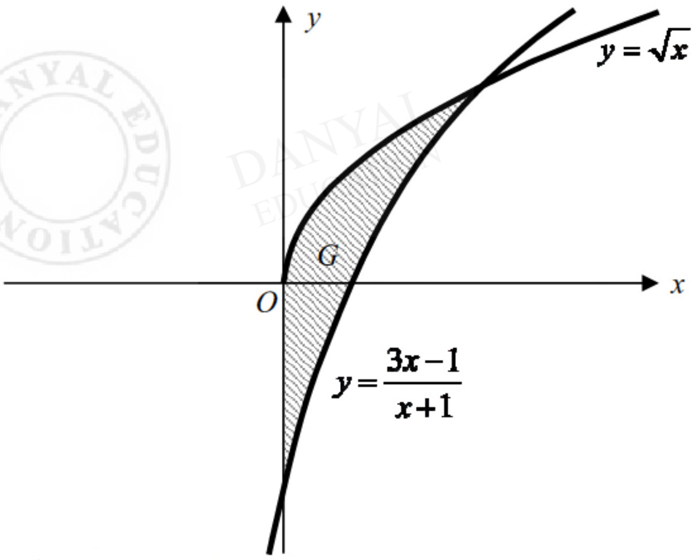
(i) Show that the area enclosed by the curve is given by

$$k\int_0^{\frac{\pi}{2}} \cos^4 t \, \mathrm{d}t,$$

where k is a constant to be determined. [3]

(ii) Hence find the exact area enclosed by the curve. [3]

(b) In the diagram, the region G is bounded by the curves $y = \dfrac{3x-1}{x+1}$, $y = \sqrt{x}$ and the y-axis.



Find the exact volume of the solid generated when G is rotated about the y-axis through 2π radians. [6]

Q2

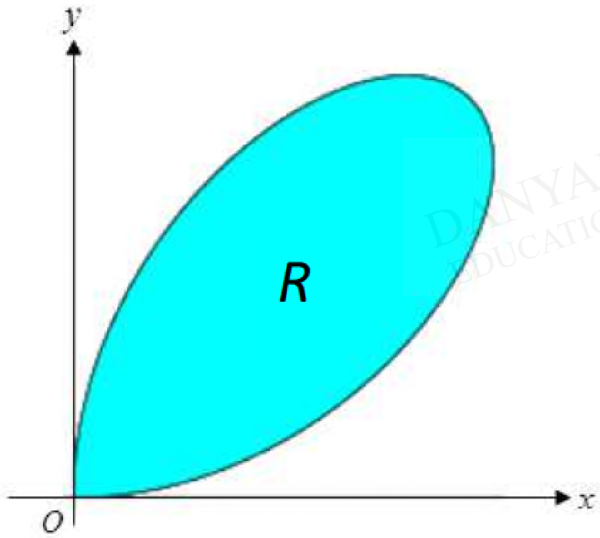
(a) Find $\int \sin(2\theta)\cos(3\theta)\,\mathrm{d}\theta$. [2]

(b) Use the substitution $\theta = \sqrt{x}$ to find the exact value of $\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \theta^3 \cos(\theta^2)\,\mathrm{d}\theta$. [5]

Q3

(a) The diagram below shows a section of *Folium of Descartes* curve which is defined parametrically by

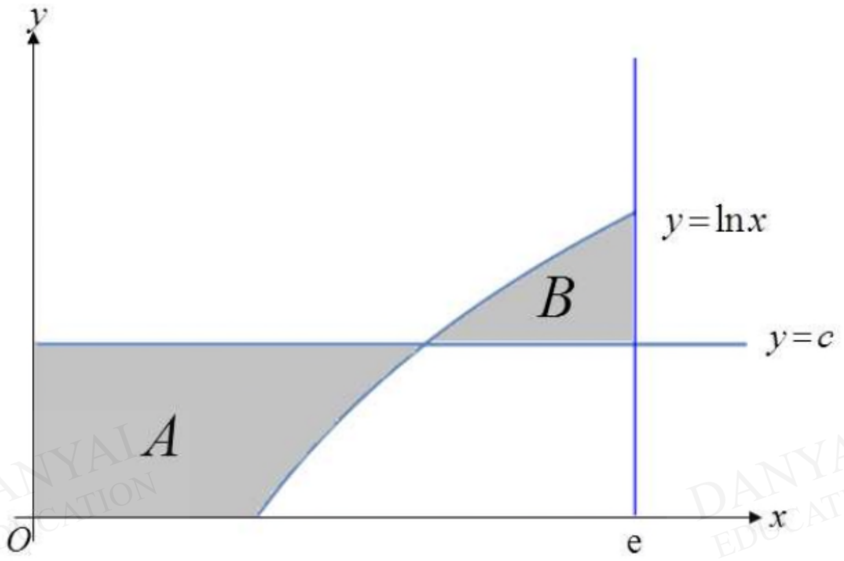
$$x = \frac{3m}{1+m^3}, \quad y = \frac{3m^2}{1+m^3}, \quad m \geq 0.$$



(i) It is known that the curve is symmetrical about the line $y = x$. Find the values of m where the curve meets the line $y = x$. [1]

(ii) Region R is the region enclosed by the curve in the first quadrant. Show that the area of R is given by $2\left(\int_0^{\frac{3}{2}} x \, dy - \frac{9}{8}\right)$, and evaluate this integral. [5]

(b) The diagram below shows a horizontal line $y = c$ intersecting the curve $y = \ln x$ at a point where the x-coordinate is such that $1 < x < e$.



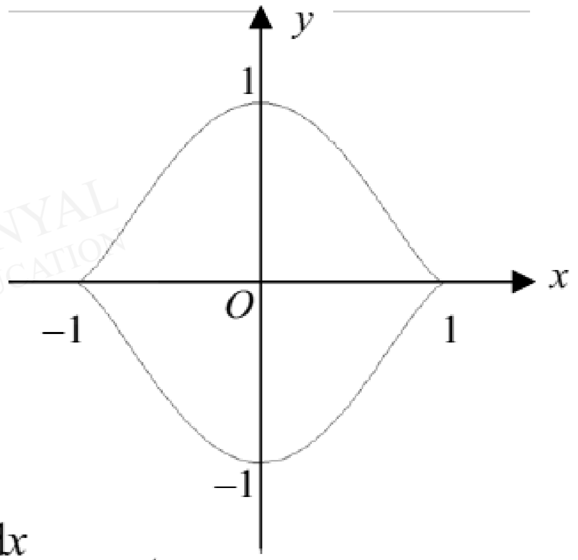
The region A is bounded by the curve, the line $y = c$, the x-axis and the y-axis while the region B is bounded by the curve and the lines $x = e$ and $y = c$. Given that the volumes of revolution when A and B are rotated completely about the y-axis are equal, show that $c = \dfrac{e^2 + 1}{2e^2}$. [6]

Answers

Integration Test 5

Q1

(a)(i)



$$x = \sin t \;\Rightarrow\; \frac{dx}{dt} = \cos t$$

When $x = 0$, $t = 0$.

When $x = 1$, $t = \frac{\pi}{2}$.

$$\text{Area} = 4\int_0^1 y \, dx$$

$$= 4\int_0^{\frac{\pi}{2}} (\cos^3 t)\cos t \, dt$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^4 t \, dt \qquad \text{(shown)}$$

$\therefore\; k = 4$

(a)(ii)

$$\text{Area} = 4\int_0^{\frac{\pi}{2}} \cos^4 t \, dt$$

$$= \int_0^{\frac{\pi}{2}} (2\cos^2 t)^2 \, dt$$

$$= \int_0^{\frac{\pi}{2}} (1+\cos 2t)^2 \ \mathrm{d}t$$

$$= \int_0^{\frac{\pi}{2}} 1+2\cos 2t+\cos^2 2t \ \mathrm{d}t$$

$$= \int_0^{\frac{\pi}{2}} 1+2\cos 2t+\frac{1+\cos 4t}{2} \ \mathrm{d}t$$

$$= \int_0^{\frac{\pi}{2}} \frac{3}{2}+2\cos 2t+\frac{\cos 4t}{2} \ \mathrm{d}t$$

$$= \left[\frac{3t}{2}+\sin 2t+\frac{\sin 4t}{8}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{3\pi}{4} \ \text{unit}^2$$

(b)

From GC, coordinates of intersection = (1, 1)

<u>Method 1</u>

$$y=\frac{3x-1}{x+1} \Rightarrow xy+y=3x-1 \Rightarrow x=\frac{1+y}{3-y}$$

Required volume

$$= \pi\int_{-1}^{1}\left(\frac{1+y}{3-y}\right)^2 \ \mathrm{d}y - \pi\int_0^1 (y^2)^2 \ \mathrm{d}y$$

$$= \pi\int_{-1}^{1}\left(\frac{4}{3-y}-1\right)^2 \ \mathrm{d}y - \pi\int_0^1 y^4 \ \mathrm{d}y$$

$$= \pi\int_{-1}^{1}\left(\frac{16}{(3-y)^2}-\frac{8}{3-y}+1\right)\mathrm{d}y - \pi\left[\frac{y^5}{5}\right]_0^1$$

$$= \pi\left[\frac{16}{3-y}+8\ln|3-y|+y\right]_{-1}^{1} - \frac{\pi}{5}$$

$$= \pi\left[8+8\ln 2+1-(4+8\ln 4-1)\right]-\frac{\pi}{5}$$

$$= \pi\left[6+8\ln 2-16\ln 2\right]-\frac{\pi}{5}$$

$$= \frac{29\pi}{5}-8\pi\ln 2 \quad \text{unit}^3$$

<u>Method 2</u>

$$y=\frac{3x-1}{x+1} \Rightarrow xy+y=3x-1 \Rightarrow x=\frac{1+y}{3-y}$$

Required volume

$$= \pi \int_{-1}^{1} \left(\frac{1+y}{3-y} \right)^2 \ \mathrm{d}y - \pi \int_0^1 (y^2)^2 \ \mathrm{d}y$$

$$= \pi \int_{-1}^{1} \frac{y^2+2y+1}{y^2-6y+9} \ \mathrm{d}y - \pi \int_0^1 y^4 \ \mathrm{d}y$$

$$= \pi \int_{-1}^{1} 1 + \frac{8y-8}{y^2-6y+9} \ \mathrm{d}y - \pi \left[\frac{y^5}{5} \right]_0^1$$

$$= \pi \left[y \right]_{-1}^{1} + 4\pi \int_{-1}^{1} \frac{2y-6}{y^2-6y+9} \ \mathrm{d}y + \pi \int_{-1}^{1} \frac{16}{(y-3)^2} \ \mathrm{d}y - \frac{\pi}{5}$$

$$= 2\pi + 4\pi \left[\ln \left| y^2-6y+9 \right| \right]_{-1}^{1} + 16\pi \left[\frac{(y-3)^{-1}}{-1} \right]_{-1}^{1} - \frac{\pi}{5}$$

$$= \frac{9\pi}{5} + 4\pi \left[\ln 4 - \ln 16 \right] + 16\pi \left[\frac{1}{3-y} \right]_{-1}^{1}$$

$$= \frac{9\pi}{5} + 4\pi \ln \frac{1}{4} + 16\pi \left[\frac{1}{2} - \frac{1}{4} \right]$$

$$= \frac{9\pi}{5} - 4\pi \ln 4 + 4\pi$$

$$= \frac{29\pi}{5} - 8\pi \ln 2 \quad \text{unit}^3$$

Q2

(a) By Factor Formula,

$$\sin(2\theta)\cos(3\theta) = \frac{1}{2}\left[\sin(5\theta)+\sin(-\theta)\right]$$

$$= \frac{1}{2}\left[\sin(5\theta)-\sin(\theta)\right]$$

$$\int \sin(2\theta)\cos(3\theta)\,\mathrm{d}\theta = \int \frac{1}{2}\left[\sin(5\theta)-\sin(\theta)\right]\mathrm{d}\theta$$

$$= \frac{1}{2}\cos\theta - \frac{1}{10}\cos(5\theta)+c$$

(b) $\theta=\sqrt{\pi} \Rightarrow \sqrt{x}=\sqrt{\pi} \Rightarrow x=\pi$

$$\theta=\sqrt{\frac{\pi}{2}} \Rightarrow \sqrt{x}=\sqrt{\frac{\pi}{2}} \Rightarrow x=\frac{\pi}{2}$$

$$\theta=\sqrt{x} \Rightarrow \frac{\mathrm{d}\theta}{\mathrm{d}x}=\frac{1}{2\sqrt{x}}.$$

$$\int_{\sqrt{\frac{\pi}{2}}}^{\sqrt{\pi}} \theta^3\cos\left(\theta^2\right)\mathrm{d}\theta$$

$$=\int_{\frac{\pi}{2}}^{\pi} x\sqrt{x}(\cos x)\left(\frac{1}{2\sqrt{x}}\right)\mathrm{d}x$$

$$=\frac{1}{2}\int_{\frac{\pi}{2}}^{\pi} x\cos x\,\mathrm{d}x$$

$u = x$	$\frac{\mathrm{d}v}{\mathrm{d}x} = \cos x$
$\frac{\mathrm{d}u}{\mathrm{d}x} = 1$	$v = \sin x$

$$=\frac{1}{2}\left[\left[x\sin x\right]_{\frac{\pi}{2}}^{\pi} - \int_{\frac{\pi}{2}}^{\pi} 1(\sin x)\,\mathrm{d}x\right]$$

$$=\frac{1}{2}\left(0-\frac{\pi}{2}+\left[\cos x\right]_{\frac{\pi}{2}}^{\pi}\right)$$

$$=\frac{1}{2}\left[-\frac{\pi}{2}+(-1-0)\right]$$

$$=-\frac{1}{2}-\frac{\pi}{4}$$

Q3

(a)(i) $x = \dfrac{3m}{1+m^3}, \quad y = \dfrac{3m^2}{1+m^3}, \quad m \geq 0$

$$y = x$$

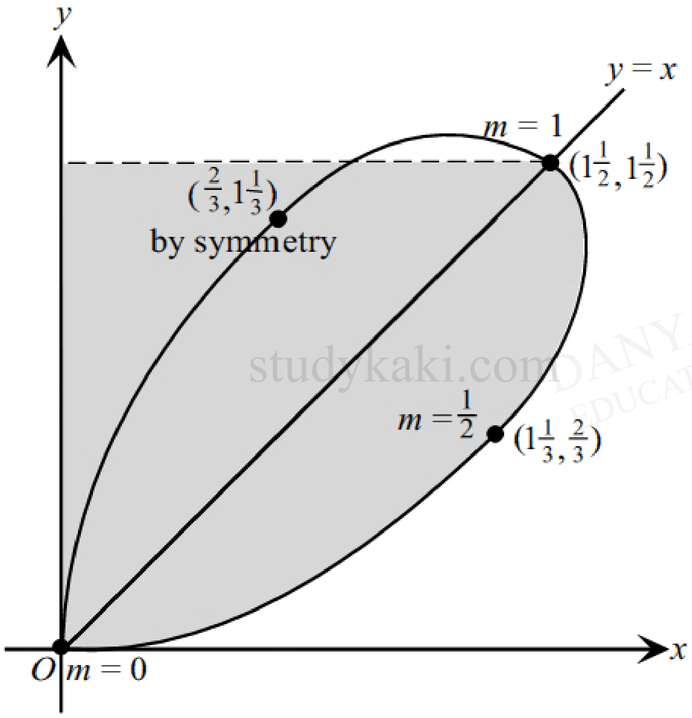
$$\frac{3m^2}{1+m^3} = \frac{3m}{1+m^3}$$

$$m(m-1) = 0$$

$$m = \underline{0 \text{ or } 1}$$

(a)(ii) When $m = 0$, $y = 0$.

When $m = 1$, $y = \dfrac{3}{1+1} = \dfrac{3}{2}$.



Notes:

Use GC to trace the path to see how m varies when the point moves along the path.

Area of (lower) half of the "leaf" is

$$\frac{1}{2}A = \int_0^{\frac{3}{2}} x\,dy - \text{area of }\Delta \qquad \left(\text{Note: } \int_0^{\frac{3}{2}} x\,dy = \text{shaded area}\right)$$

$$A = 2\left[\int_0^{\frac{3}{2}} x\,dy - \frac{1}{2}\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)\right]$$

$$= 2\left(\int_0^{\frac{3}{2}} x\,dy - \frac{9}{8}\right) \qquad \text{(Shown)}$$

$$2\left(\int_0^{\frac{3}{2}} x\,dy - \frac{9}{8}\right) = 2\int_0^1 \frac{3m}{1+m^3}\left[\frac{6m\left(1+m^3\right)-3m^2\left(3m^2\right)}{\left(1+m^3\right)^2}\right]dm - \frac{9}{4}$$

$$= 2\int_0^1 \frac{3m\left(6m-3m^4\right)}{\left(1+m^3\right)^3}dm - \frac{9}{4}$$

$$= \frac{15}{4} - \frac{9}{4} \qquad \text{(by GC)}$$

$$= \underline{\underline{\frac{3}{2}}}$$

(b)

$$y = \ln x$$
$$x = e^y$$

$$V_A = \pi\int_0^c \left(e^y\right)^2 dy$$
$$= \pi\int_0^c e^{2y}\,dy$$
$$= \pi\left[\frac{1}{2}e^{2y}\right]_0^c$$
$$= \frac{\pi}{2}\left(e^{2c}-1\right)$$

$$V_B = (1-c)\pi e^2 - \pi\int_c^1 \left(e^y\right)^2 dy \quad \text{or} \quad \pi\int_c^1 \left[e^2 - \left(e^y\right)^2\right]dy$$
$$= \pi(1-c)e^2 - \pi\left[\frac{1}{2}e^{2y}\right]_c^1$$
$$= \pi(1-c)e^2 - \frac{\pi}{2}\left(e^2 - e^{2c}\right)$$

$$V_A = V_B$$
$$\frac{\pi}{2}\left(e^{2c}-1\right) = \pi(1-c)e^2 - \frac{\pi}{2}\left(e^2-e^{2c}\right)$$
$$e^{2c}-1 = 2e^2(1-c) - e^2 + e^{2c}$$
$$= 2e^2 - 2ce^2 - e^2 + e^{2c}$$
$$2ce^2 = e^2+1$$
$$c = \frac{e^2+1}{2e^2} \qquad \text{(Shown)}$$