A Level H2 Math

Integration Test 4

Q1

(i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\tan^2 x$. Hence evaluate $\int_0^{\frac{1}{4}\pi} \sec^2 x \tan x \, \mathrm{e}^{\tan^2 x} \, \mathrm{d}x$, leaving your answer in exact form. [3]

(ii) By expressing $1+72x-32x^3$ as $1+mx(9-4x^2)$ where m is a constant, find $\int \frac{1+72x-32x^3}{\sqrt{(9-4x^2)}} \, \mathrm{d}x$. [2]

Q2

The curve C with equation $y=\frac{x^2+(a-1)x-a-1}{x-1}$, where a is a constant, has the oblique asymptote $y=x+1$.

(i) Show that $a = 1$. Hence sketch C, giving the equations of any asymptotes and the exact coordinates of any points of intersection with the axes. [3]

(ii) The region bounded by C for $x>1$ and the lines $y=x+1$, $y=2$ and $y=4$ is rotated through 2π radians about the line $x=1$. By considering a translation of C, or otherwise, find the volume of revolution formed. [5]

Q3

By writing $\sec^3 x = \sec x \sec^2 x$, find $\int \sec^3 x \, \mathrm{d}x$.

Hence find the exact value of $\int_0^{\tan^{-1} 2} \sec^3 x \, \mathrm{d}x$. [6]

Answers

Integration Test 4

Q1

(i) $$\int_0^{\frac{\pi}{4}} \sec^2 x \tan x \, e^{\tan^2 x} \, dx = \frac{1}{2}\int_0^{\frac{\pi}{4}} 2\sec^2 x \tan x \, e^{\tan^2 x} \, dx$$

$$= \frac{1}{2}\left[e^{\tan^2 x}\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(e^{\tan^2 \frac{\pi}{4}} - e^{\tan^2 0}\right)$$

$$= \frac{1}{2}(e-1)$$

(ii) $$\int \frac{1+72x-32x^3}{\sqrt{9-4x^2}} \, dx = \int \frac{1+8x\left(9-4x^2\right)}{\sqrt{9-4x^2}} \, dx$$

$$= \int \frac{1}{\sqrt{9-4x^2}} + 8x\left(9-4x^2\right)^{\frac{1}{2}} \, dx$$

$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) - \frac{2}{3}\left(9-4x^2\right)^{\frac{3}{2}} + C$$

Q2

(i) $$y=\frac{x^2+(a-1)x-a-1}{x-1}$$
$$=\frac{(x+a)(x-1)-1}{x-1}$$
$$=(x+a)-\frac{1}{x-1}$$

Given that oblique asymptote is $y=x+1$, $\therefore a=1$ (shown)

<u>Alternative</u>

Let $\frac{x^2+(a-1)x-a-1}{x-1}=(x+1)+\frac{b}{x-1}$

$\Rightarrow x^2+(a-1)x-a-1=x^2-1+b$

Comparing coeff of x:

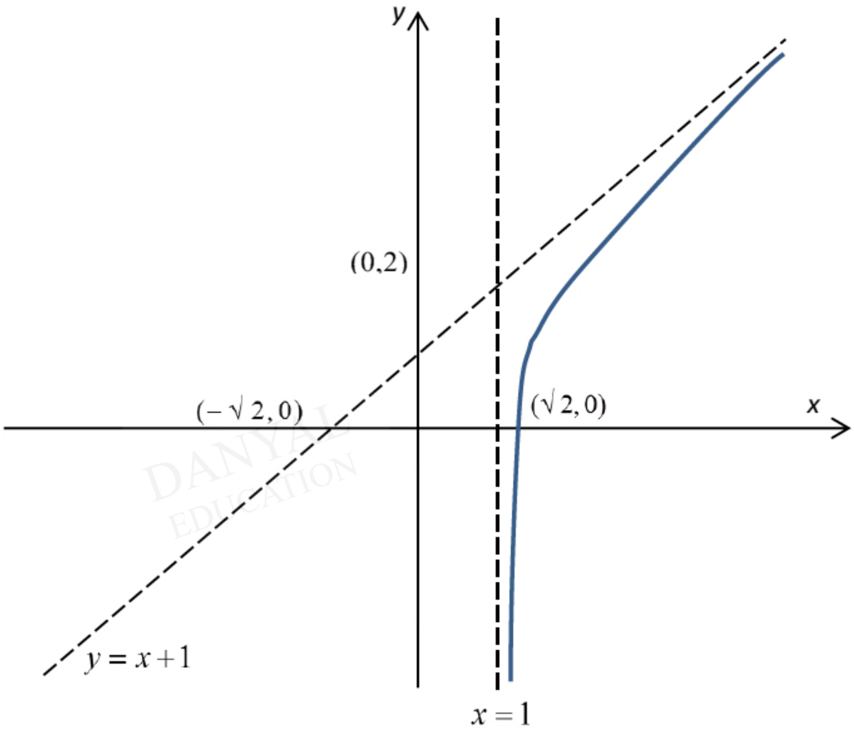
$a-1=0$

$\therefore a=1$ (shown) and $b=-1$

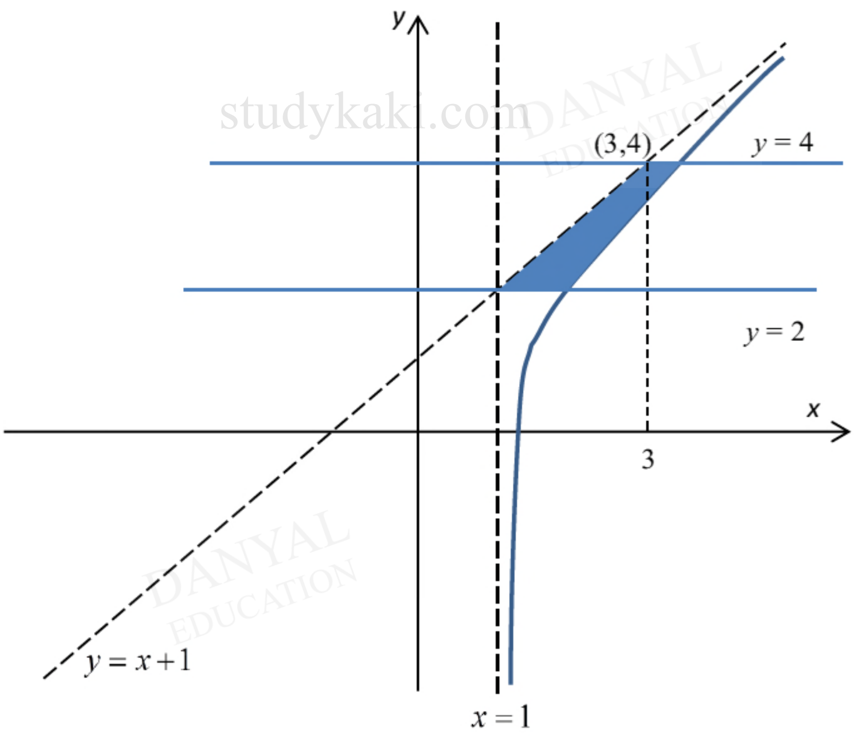
$\therefore y=(x+1)-\frac{1}{x-1}=\frac{x^2-2}{x-1}$

HA: $x=1$

OA: $y=x+1$ (given)



(ii)



$$y=\frac{x^2-2}{x-1}\xrightarrow{\text{replace } x \text{ with } (x+1)} y=\frac{(x+1)^2-2}{x}$$

$$y=\frac{(x+1)^2-2}{x}$$

$$xy=x^2+2x-1$$

$$x^2+(2-y)x-1=0$$

$$x=\frac{-(2-y)\pm\sqrt{(2-y)^2+4(1)(1)}}{2}$$

$$\therefore x=\frac{(y-2)+\sqrt{y^2-4y+8}}{2}$$ **(reject -ve root)**

$$\text{Volume}=\pi\int_2^4\left(\frac{(y-2)+\sqrt{y^2-4y+8}}{2}\right)^2\,\mathrm{d}y-\frac{1}{3}\pi(2)^2(2)$$

$$=9.75\ \text{units}^3\ (3\ \text{s.f})$$

Q3

$$u = \sec x \Rightarrow u' = \sec x \tan x$$
$$v' = \sec^2 x \Rightarrow v = \tan x$$

$$\int \sec^3 x \, dx$$
$$= \int \sec x \sec^2 x \, dx$$
$$= \sec x \tan x - \int \sec x \tan^2 x \, dx$$
$$= \sec x \tan x - \int \sec x(\sec^2 x - 1) \, dx$$
$$= \sec x \tan x - \int \sec^3 x - \sec x \, dx$$
$$= \sec x \tan x - \int \sec^3 x \, dx + \ln|\sec x + \tan x|$$
$$2\int \sec^3 x \, dx = \sec x \tan x + \ln|\sec x + \tan x|$$
$$\int \sec^3 x \, dx = \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) + C$$

$$\int_0^{\tan^{-1} 2} \sec^3 x \, dx$$
$$= \frac{1}{2}\left[\sec x \tan x + \ln|\sec x + \tan x|\right]_0^{\tan^{-1} 2}$$
$$= \frac{1}{2}\left[\sqrt{5} \times 2 + \ln(\sqrt{5} + 2)\right]$$
$$= \sqrt{5} + \frac{1}{2}\ln(\sqrt{5} + 2)$$