A Level H2 Math

Integration Test 3

Q1

The region bounded by the curve $y=\dfrac{1}{\sqrt{x}-2}$, the x-axis and the lines $x=9$ and $x=16$ is rotated through 2π radians about the x-axis. Use the substitution $t=\sqrt{x}$ to find the exact volume of the solid obtained. [6]

Q2

(a) Find $\int \mathrm{e}^{x}\cos(2x)\,\mathrm{d}x$. [3]

(b) The curve C has parametric equations

$$x=t-\mathrm{e}^{t},\quad y=3\cos^{2}t-1,\quad \text{for } 0<t<\pi.$$

(i) Use differentiation to find the exact x-coordinate of any turning point and determine the nature of the turning point. [3]

(ii) Find the exact area of the region bounded by the curve C and the line $y=2$, expressing your answer in the form $a\pi+b+c\mathrm{e}^{\pi}$, where a, b and c are rational numbers to be determined. [5]

Q3

The parametric equations of the curve C are

$$x=2\sec t \text{ and } y=3\tan t\text{, where } -\pi<t\le\pi,\ t\neq\pm\frac{\pi}{2}.$$

(i) Write down the Cartesian equation of C. [1]

(ii) Sketch the curve C, stating the equations of the asymptotes and the coordinates of the points where C crosses the axes, if any. [2]

(iii) The line $y=\sqrt{3}x+k$, where $k<0$, is a tangent to C. Show that $k=-\sqrt{3}$. [3]

The region bounded by this tangent, the curve C and the x-axis is rotated completely about the x-axis. Calculate the volume obtained. [4]

Answers

Integration Test 3

Q1

Method ①:

Volume

$$= \pi \int_9^{16} \left(\frac{1}{\sqrt{x}-2} \right)^2 \, dx$$

$$= \pi \int_3^4 \left(\frac{1}{t-2} \right)^2 (2t) \, dt$$

$$= \pi \int_3^4 \frac{2t}{t^2-4t+4} \, dt$$

$$= \pi \int_3^4 \frac{2t-4}{t^2-4t+4} + \frac{4}{(t-2)^2} \, dt$$

$$= \pi \left[\ln\left|t^2-4t+4\right| \right]_3^4 + \pi \int_3^4 4(t-2)^{-2} \, dt$$

$$= \pi \left[\ln\left|t^2-4t+4\right| + 4\frac{(t-2)^{-1}}{-1} \right]_3^4$$

$$= \pi \left[\ln\left|t^2-4t+4\right| - \frac{4}{(t-2)} \right]_3^4$$

$$= \pi\left[(\ln 4 - 2) - (\ln 1 - 4) \right]$$

$$= \pi(\ln 4 + 2) \text{ units}^3$$

Substitution:

$t = \sqrt{x}$

$t^2 = x$

$2t = \frac{dx}{dt}$

When $x = 9, t = 3$

When $x = 16, t = 4$

Most candidates were able to setup the correct integral for the volume of revolution. However, many failed make the correct substitution of dx by $\frac{dx}{dt}dt$ and thus $2tdt$.

Another group of students forgot to change the upper and lower limits to the respective values of t when the variable was changed.

Many students were also stuck at the integration of $\frac{4}{(t-2)^2}$ as it is not very easy for those who don't practise much to identify the fraction as a power function of power -2.

Method ②:

Volume

$$= \pi \int_9^{16} \left(\frac{1}{\sqrt{x}-2} \right)^2 \, dx$$

$$= \pi \int_3^4 \left(\frac{1}{t-2} \right)^2 (2t) \, dt$$

$$= \pi \int_3^4 \frac{2t}{(t-2)^2} \, dt$$

$$= \pi \int_3^4 \frac{2}{(t-2)} + \frac{4}{(t-2)^2} \, dt$$

$$= \pi \left[2\ln|t-2| \right]_3^4 + \pi \int_3^4 4(t-2)^{-2} \, dt$$

$$= \pi \left[2\ln|t-2| + 4\frac{(t-2)^{-1}}{-1} \right]_3^4$$

$$= \pi \left[2\ln|t-2| - \frac{4}{(t-2)} \right]_3^4$$

$$= \pi\left[(2\ln 2 - 2) - (2\ln 1 - 4) \right]$$

$$= \pi(2\ln 2 + 2) \text{ units}^3$$

Substitution:

$t = \sqrt{x}$

$t^2 = x$

$2t = \frac{dx}{dt}$

When $x = 9, t = 3$

When $x = 16, t = 4$

Q2

(a)

$$\int e^x \cos 2x \, dx = e^x \cos 2x + 2\int e^x \sin 2x \, dx$$

$$= e^x \cos 2x + 2\left(e^x \sin 2x - 2\int e^x \cos 2x \, dx\right)$$

$$5\int e^x \cos 2x \, dx = e^x \cos 2x + 2e^x \sin 2x$$

$$\therefore \int e^x \cos 2x \, dx = \frac{e^x(\cos 2x + 2\sin 2x)}{5} + c$$

(b)(i)

$$x = t - e^t, \quad y = 3\cos^2 t - 1$$

$$\frac{dx}{dt} = 1 - e^t, \quad \frac{dy}{dt} = 6\cos t(-\sin t) = -3\sin(2t)$$

$$\frac{dy}{dx} = \frac{-3\sin(2t)}{1-e^t}$$

$$\frac{dy}{dx} = 0 \Rightarrow \sin(2t) = 0$$

$$\Rightarrow 2t = 0 \text{ (N.A.) or } 2t = \pi \text{ or } 2t = 2\pi \text{(N.A.)}$$

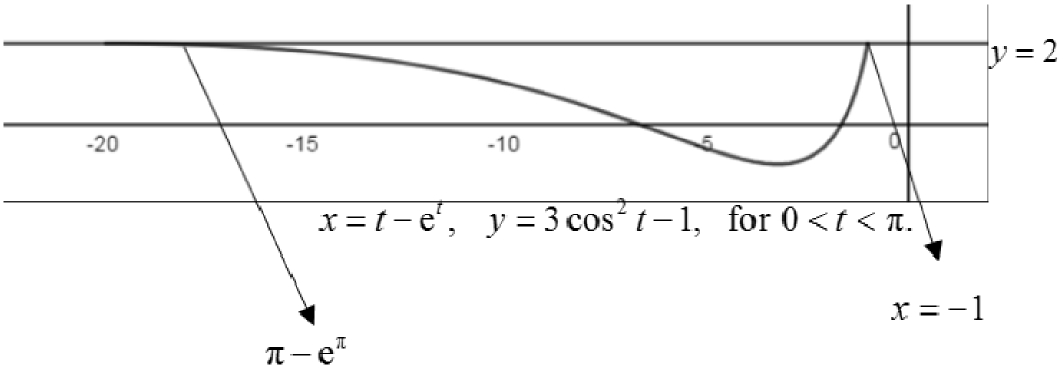
$$\Rightarrow t = \frac{\pi}{2}$$

t	1.6	$\frac{\pi}{2}$	1.5
x	-3.35303	-3.2396811	-2.981689
$\frac{dy}{dx}$	-0.0443	0	0.122

NB: t increases as x decreases.

Hence x-coordinate of the minimum point at is $\frac{\pi}{2} - e^{\frac{\pi}{2}}$.

(b)(ii)



When $y = 2$,

$2 = 3\cos^2 t - 1$

$\Rightarrow \cos t = \pm 1$

$\Rightarrow t = 0, \pi$

When $t = 0, x = 0 - e^0 = -1$

When $t = \pi, x = \pi - e^{\pi} = -19.9991$

Area required

$$= \int_{\pi - e^{\pi}}^{-1} (2 - y)\, dx$$

$$= \int_{\pi}^{0} \left(2 - \left(3\cos^2 t - 1\right)\right)\left(1 - e^t\right) dt$$

$$= \int_{\pi}^{0} \left(3 - 3\cos^2 t\right)\left(1 - e^t\right) dt$$

$$= 3\int_{\pi}^{0} \left(1 - \cos^2 t\right)\left(1 - e^t\right) dt$$

$$= 3\int_{\pi}^{0} \left(1 - \frac{\cos 2t + 1}{2}\right)\left(1 - e^t\right) dt$$

$$= 3\int_{\pi}^{0} \left(\frac{1}{2} - \frac{\cos 2t}{2}\right)\left(1 - e^t\right) dt$$

$$= \frac{3}{2}\int_{\pi}^{0} \left(1 - \cos 2t\right)\left(1 - e^t\right) dt$$

$$= \frac{3}{2}\int_{\pi}^{0} \left(1 - \cos 2t - e^t + e^t \cos 2t\right) dt$$

$$= \frac{3}{2}\left[t - \frac{\sin 2t}{2} - e^t + \frac{e^t\left(\cos 2t + 2\sin 2t\right)}{5}\right]_{\pi}^{0}$$

$$= \frac{3}{2}\left[-\frac{4}{5} - \pi + \frac{4e^{\pi}}{5}\right]$$

$$= -\frac{3}{2}\pi - \frac{6}{5} + \frac{6}{5}e^{\pi}, \text{ where } a = -\frac{3}{2},\ b = -\frac{6}{5},\ c = \frac{6}{5}$$

Q3

(i)

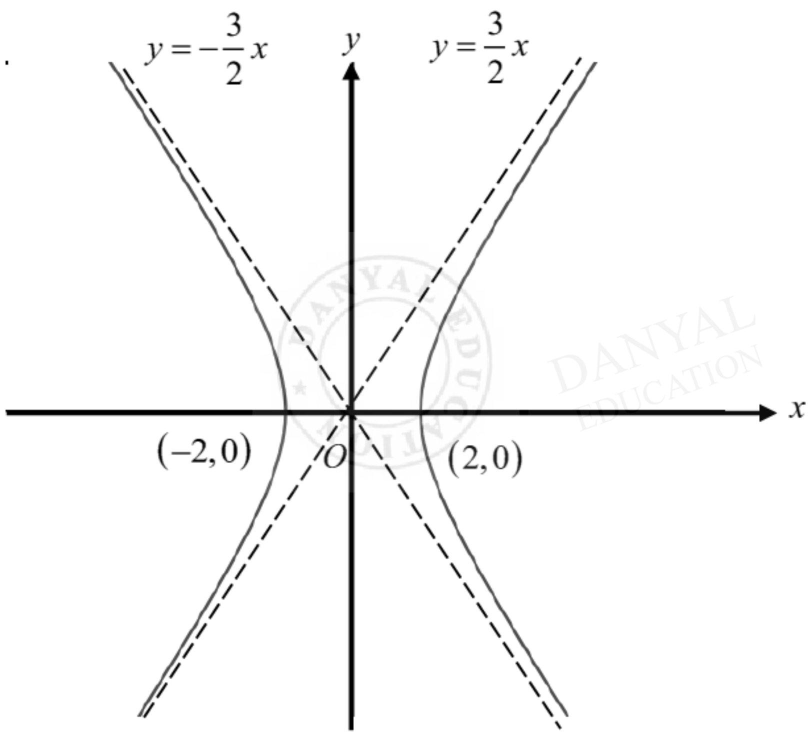
$x = 2\sec t$ and $y = 3\tan t$

$1 + \tan^2 t = \sec^2 t$

$\Rightarrow 1 + \dfrac{y^2}{9} = \dfrac{x^2}{4}$

$\Rightarrow \dfrac{x^2}{4} - \dfrac{y^2}{9} = 1$

(ii)



(iii)

Method 1

$$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 3\sec^2 t \cdot \frac{1}{2\sec t \tan t} = 1.5\operatorname{cosec} t$$

$1.5\operatorname{cosec} t = \sqrt{3}$

$t = \frac{\pi}{3}$ or $\frac{2\pi}{3}$

When $t = \frac{\pi}{3}$,

$x = 2\sec\frac{\pi}{3} = 4$, $y = 3\tan\frac{\pi}{3} = 3\sqrt{3}$.

Equation of tangent:

$y - 3\sqrt{3} = \sqrt{3}(x-4)$

$\Rightarrow y = \sqrt{3}x - 4\sqrt{3} + 3\sqrt{3}$

$\Rightarrow y = \sqrt{3}x - \sqrt{3}$

$\therefore k = -\sqrt{3}$ (Shown)

When $t = \dfrac{2\pi}{3}$,

$x = 2\sec\dfrac{2\pi}{3} = -4$, $y = 3\tan\dfrac{2\pi}{3} = -3\sqrt{3}$.

Equation of tangent:

$$y + 3\sqrt{3} = \sqrt{3}(x+4)$$
$$\Rightarrow y = \sqrt{3}x + 4\sqrt{3} - 3\sqrt{3}$$
$$\Rightarrow y = \sqrt{3}x + \sqrt{3}$$
$$\therefore k = \sqrt{3} \text{ (N.A. } \because k < 0)$$

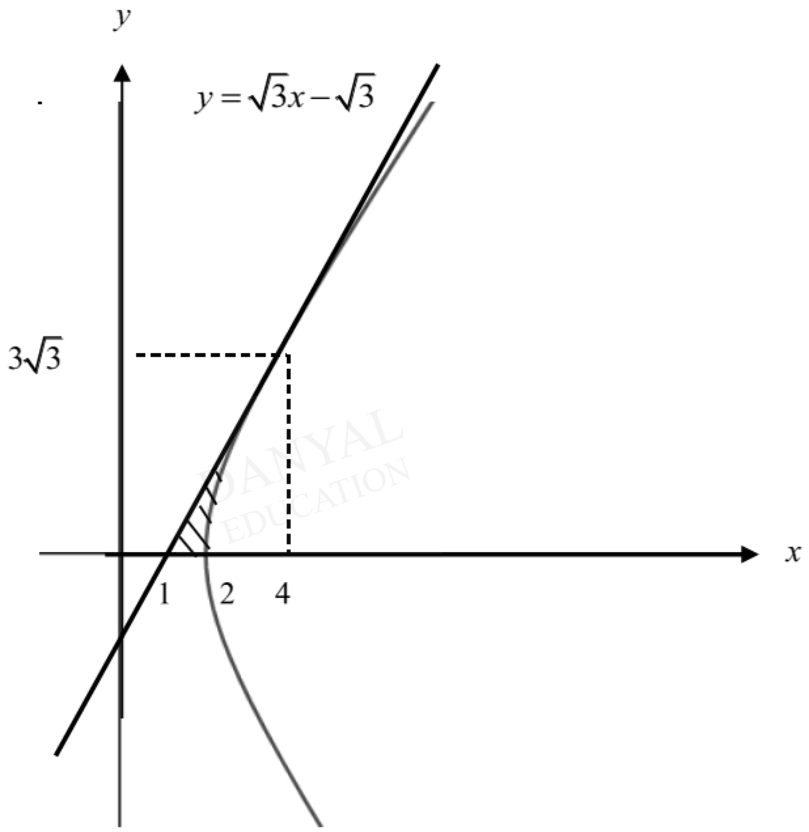
Method 2

$$\frac{x^2}{4} - \frac{\left(\sqrt{3}x + k\right)^2}{9} = 1$$
$$\Rightarrow -3x^2 - \left(8\sqrt{3}k\right)x - \left(36 + 4k^2\right) = 0$$

Since the line $y = \sqrt{3}x + k$, where $k < 0$, is a tangent to C, there should be repeated roots.

Thus,

$$\left(8\sqrt{3}k\right)^2 - 4(-3)\left(-36 - 4k^2\right) = 0$$
$$\Rightarrow 192k^2 - 432 - 48k^2 = 0$$
$$\Rightarrow 144k^2 = 432$$
$$\Rightarrow k^2 = 3$$
$$\Rightarrow k = \sqrt{3} \text{ (N.A. } \because k < 0) \text{ or } k = -\sqrt{3} \text{ (Shown)}$$



For $k=-\sqrt{3}$,

$-3x^2-\left(8\sqrt{3}\left(-\sqrt{3}\right)\right)x-\left(36+4(3)\right)=0$

$3x^2-24x+48=0$

$x^2-8x+16=0$

$(x-4)^2=0$

$\Rightarrow x=4$

$y^2=9\left(\frac{x^2}{4}-1\right)$

When $x=4$, $y=\sqrt{27}=3\sqrt{3}\ (y>0)$

$=\frac{1}{3}\pi\left(3\sqrt{3}\right)^2(3)-\pi\int_2^4 y^2\ \mathrm{d}x$

$=27\pi-9\pi\int_2^4\left(\frac{x^2}{4}-1\right)\ \mathrm{d}x$

$=27\pi-9\pi\left(\frac{8}{3}\right)$

$=3\pi$

$=9.42\ (3\text{ sf})$