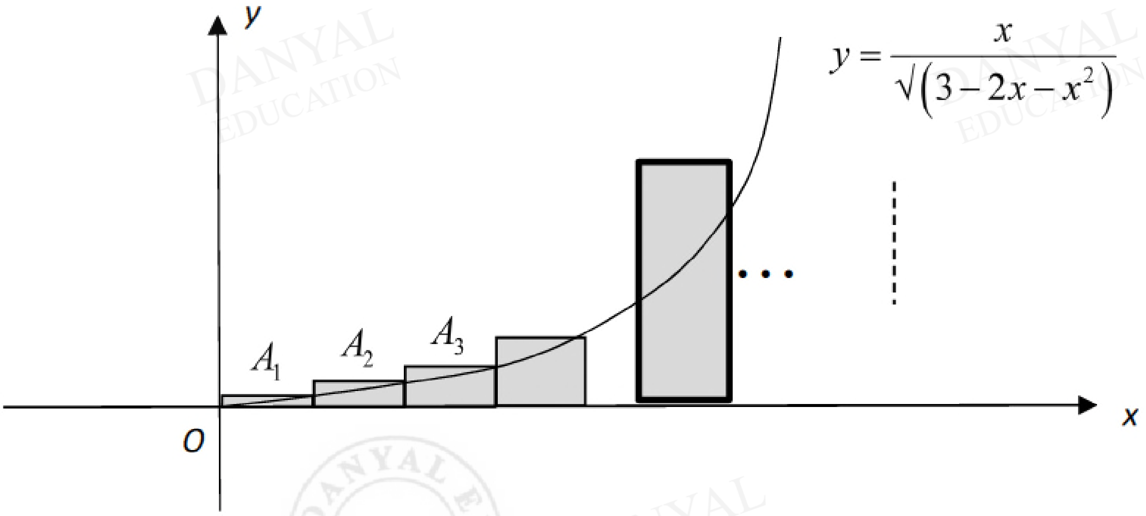
A Level H2 Math

Integration Test 2

Q1

(a) Find $\int e^x \sin x \, dx$. [3]

(b)



The diagram shows the curve with eq$k$uation $y = \dfrac{x}{\sqrt{(3-2x-x^2)}}$ for $0 \le x < 1$.

The region bounded by the curve, the x-axis and the line $x = k$, $0 < k < 1$ is denoted by S. It is given that n rectangles of equal width are drawn between $x = 0$ and $x = k$.

(i) Show that the area of the first rectangle, $A_1 = \dfrac{k^2}{n\sqrt{3n^2 - 2nk - k^2}}$. [1]

(ii) Show that the total area of all the n rectangles is

$$\sum_{r=1}^{n} \frac{rk^2}{n\sqrt{(3n^2 - anrk - br^2k^2)}},$$

where a and b are constants to be determined. [2]

It is now given that $k = (\sqrt{3}) - 1$.

(iii) Use integration to find the actual area of region S. Hence state the exact value of

$$\sum_{r=1}^{\infty} \frac{rk^2}{n\sqrt{(3n^2 - anrk - br^2k^2)}}.$$ [6]

Q2

(i) By using the substitution $t = 3\sec\theta$, find $\int \frac{\sqrt{t^2-9}}{t}\,dt$. [4]

(ii) The curve C is defined by the parametric equations

$$x = \ln t, \quad y = \sqrt{t^2-9}, \text{ where } t \geq 3.$$

Find the exact value of the area of the region bounded by C, the line $x = \ln 6$ and the x-axis. [4]

Q3

(a) Show that $\int \sqrt{5-x^2}\,dx = \frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$. [4]

(b) **(i)** Let C be the curve $y^4 + x^2 = 5$. The x-coordinate of the point P on C is 1 and the y-coordinate of the point P on C is positive. Show that the gradient of the normal to C at the point P is $4\sqrt{2}$. Hence find the equation of the normal to C at the point P in exact form. [4]

(ii) The region R is bounded by the curve C. The solid S is formed by rotating the region R through π radians about the x-axis. Using part **(a)**, find the exact volume of the solid S in terms of π. [3]

Answers

Integration Test 2

Q1

(a)

$$\int e^x \sin x \, dx$$

$$= e^x \sin x - \int e^x \cos x \, dx$$

$$= e^x \sin x - \left[e^x \cos x + \int e^x \sin x \, dx \right]$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx$$

Hence,

$$\int e^x \sin x \, dx = e^x \sin x - e^x \cos x - \int e^x \sin x \, dx + C$$

$$2 \int e^x \sin x \, dx = e^x \sin x - e^x \cos x + C$$

$$\int e^x \sin x \, dx = \frac{1}{2}\left(e^x \sin x - e^x \cos x \right) + D$$

(b)(i)

Area of first rectangle, $x = \frac{k}{n}$:

$$A_1 = \frac{k/n}{\sqrt{3 - 2\left(k/n\right) - \left(k/n\right)^2}} \cdot \frac{k}{n} = \frac{k^2/n^2}{\sqrt{\frac{3n^2 - 2nk - k^2}{n^2}}} = \frac{k^2}{n\sqrt{3n^2 - 2nk - k^2}}$$

(b)(ii)

Area of second rectangle,

$$x = \frac{2k}{n} : A_2 = \frac{2k/n}{\sqrt{3 - 2\left(2k/n\right) - \left(2k/n\right)^2}} \cdot \frac{k}{n} = \frac{2k^2}{n\sqrt{3n^2 - 2n(2k) - (2k)^2}}$$

Area of third rectangle,

$$x = \frac{3k}{n} : A_3 = \frac{3k/n}{\sqrt{3 - 2\left(3k/n\right) - \left(3k/n\right)^2}} \cdot \frac{k}{n} = \frac{3k^2}{n\sqrt{3n^2 - 2n(3k) - (3k)^2}}$$

By observation, combined area of n rectangles:

$$A = \sum_{r=1}^{n} \frac{rk^2}{n\sqrt{3n^2 - 2nrk - r^2k^2}} ,$$

where $a = 2$ *and* $b = 1$

(b)(iii)

$$\sum_{r=1}^{\infty} \frac{rk^2}{n\sqrt{\left(3n^2 - anrk - br^2k^2\right)}}$$

$=$ Area under curve from $x = 0$ to $x = \sqrt{3} - 1$

$$= \int_0^{\sqrt{3}-1} \frac{x}{\sqrt{3-2x-x^2}}\, dx$$

$$= \int_0^{\sqrt{3}-1} \frac{-\tfrac{1}{2}(-2-2x) - 1}{\sqrt{3-2x-x^2}}\, dx$$

$$= -\frac{1}{2}\int_0^{\sqrt{3}-1} \frac{-2-2x}{\sqrt{3-2x-x^2}}\, dx - \int_0^{\sqrt{3}-1} \frac{1}{\sqrt{4-(x+1)^2}}\, dx$$

$$= -\frac{1}{2}\left[\frac{\sqrt{3-2x-x^2}}{\tfrac{1}{2}}\right]_0^{\sqrt{3}-1} - \left[\sin^{-1}\left(\frac{x+1}{2}\right)\right]_0^{\sqrt{3}-1}$$

$$= -\left[\sqrt{3-2x-x^2}\right]_0^{\sqrt{3}-1} - \left[\sin^{-1}\left(\frac{x+1}{2}\right)\right]_0^{\sqrt{3}-1}$$

$$= -\left[1-\sqrt{3}\right] - \left[\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\frac{1}{2}\right]$$

$$= \sqrt{3} - 1 - \frac{\pi}{3} + \frac{\pi}{6}$$

$$= \sqrt{3} - 1 - \frac{\pi}{6} \qquad \text{(exact)}$$

Q2

(i)

Given $t = 3\sec\theta \Rightarrow \dfrac{\mathrm{d}t}{\mathrm{d}\theta} = 3\sec\theta\tan\theta$

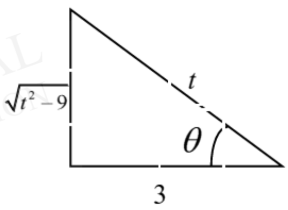
$$\int \frac{\sqrt{t^2-9}}{t}\,\mathrm{d}t$$

$$=\int \sqrt{9\sec^2\theta-9}\left(\frac{1}{3\sec\theta}\right)(3\sec\theta\tan\theta)\,\mathrm{d}\theta$$

$$=3\int \tan^2\theta\,\mathrm{d}\theta$$

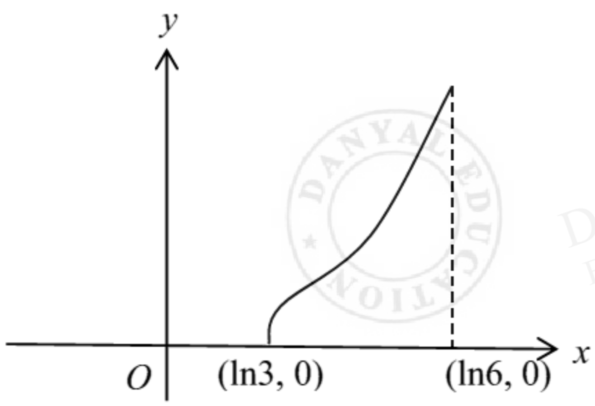
$$=3\int \sec^2\theta-1\,\mathrm{d}\theta$$

$$=3(\tan\theta-\theta)+c$$



$$=3\left(\frac{\sqrt{t^2-9}}{3}-\cos^{-1}\left(\frac{3}{t}\right)\right)+c$$

(ii)



$$\frac{\mathrm{d}x}{\mathrm{d}t}=\frac{1}{t}$$

$$\text{Area of } S = \int_{\ln 3}^{\ln 6} y\,\mathrm{d}x$$

$$=\int_3^6 \sqrt{t^2-9}\left(\frac{1}{t}\right)\mathrm{d}t$$

$$=\int_3^6 \frac{\sqrt{t^2-9}}{t}\,\mathrm{d}t$$

$$=3\left[\frac{\sqrt{t^2-9}}{3}-\cos^{-1}\left(\frac{3}{t}\right)\right]_3^6$$

$$=3\left(\frac{\sqrt{27}}{3}-\frac{\pi}{3}\right)$$

$$=3\sqrt{3}-\pi$$

Q3

(a)

$$\int \sqrt{5-x^2}\ \mathrm{d}x = x\sqrt{5-x^2} - \int \frac{-x^2}{\sqrt{5-x^2}}\ \mathrm{d}x$$

$$= x\sqrt{5-x^2} - \int \frac{\left(5-x^2\right)-5}{\sqrt{5-x^2}}\ \mathrm{d}x$$

$$= x\sqrt{5-x^2} - \int \sqrt{5-x^2}\ \mathrm{d}x + 5\int \frac{1}{\sqrt{5-x^2}}\ \mathrm{d}x$$

$$= x\sqrt{5-x^2} - \int \sqrt{5-x^2}\ \mathrm{d}x + 5\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)$$

$$\Rightarrow 2\int \sqrt{5-x^2}\ \mathrm{d}x = x\sqrt{5-x^2} + 5\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c'$$

$$\Rightarrow \int \sqrt{5-x^2}\ \mathrm{d}x = \frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right) + c$$

(bi)

$y^4 + x^2 = 5$

Differentiating wrt x,

$4y^3\frac{\mathrm{d}y}{\mathrm{d}x} = -2x$

When $x = 1$, $y^4 = 4$

$y = \pm\sqrt{2}$

At $\left(1, \sqrt{2}\right)$, $4\left(\sqrt{2}\right)^3\frac{\mathrm{d}y}{\mathrm{d}x} = -2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{4\sqrt{2}}$

Gradient of normal at $\left(1, \sqrt{2}\right)$

$= -\frac{1}{-\frac{1}{4\sqrt{2}}}$

$= 4\sqrt{2}$ (shown)

Equation of normal: $y - \sqrt{2} = 4\sqrt{2}\left(x-1\right)$

$y = 4\sqrt{2}x - 3\sqrt{2}$

(bii)

Volume of S= $\pi\int_{-\sqrt{5}}^{\sqrt{5}} y^2\ \mathrm{d}x = 2\pi\int_0^{\sqrt{5}} \sqrt{5-x^2}\ \mathrm{d}x$

$$= 2\pi\left[\frac{x}{2}\sqrt{5-x^2} + \frac{5}{2}\sin^{-1}\left(\frac{x}{\sqrt{5}}\right)\right]_0^{\sqrt{5}}$$

$$= 2\pi\left[\frac{5}{2}\left(\frac{\pi}{2}\right) - 0\right] = \frac{5}{2}\pi^2$$