A Level H2 Math

Integration Test 12

Q1
Find

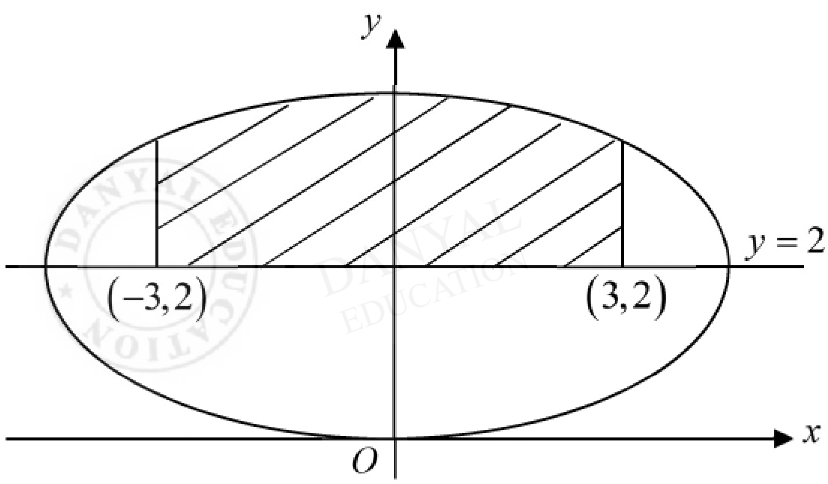
(a) $\int \cos(\ln x)\, \mathrm{d}x$, [3]

(b) $\int \dfrac{1-2x}{2x^2+1}\, \mathrm{d}x$. [3]

Q2

(a) By using the substitution $x = 3\sec\theta$, evaluate $\int_{3\sqrt{2}}^{6} \dfrac{3x+1}{\sqrt{x^2-9}}\, \mathrm{d}x$ exactly. [5]

(b)



The diagram shows an ellipse with equation $\dfrac{x^2}{16} + \dfrac{(y-2)^2}{4} = 1$.

(i) Find the area of the shaded region, giving your answer correct to 3 decimal places. [2]

(ii) Find the exact volume of the solid generated when the shaded region is rotated 180° about the y-axis. [4]

Q3

A curve C has parametric equations

$$x = t^2, \quad y = t - t^3, \; t \leq 0.$$

(i) The point P on the curve has parameter p. Show that the equation of the tangent at P is $2py = x\left(1 - 3p^2\right) + p^2 + p^4$. [3]

(ii) If the tangent at P passes through the point (6, 5), find the possible coordinates of P. [3]

(iii) Find the area of the region bounded by C and the x-axis. [3]

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Answers

Q1

4(a)	$\int \cos(\ln x)\ \mathrm{d}x = x\cos(\ln x) - \int -x\sin(\ln x).\frac{1}{x}\ \mathrm{d}x$ $= x\cos(\ln x) + \int \sin(\ln x)\ \mathrm{d}x$ $= x\cos(\ln x) + x\sin(\ln x) - \int \cos(\ln x)\ \mathrm{d}x$ $2\int \cos(\ln x)\ \mathrm{d}x = x\cos(\ln x) + x\sin(\ln x) + \text{constant}$ $\int \cos(\ln x)\ \mathrm{d}x = \frac{1}{2}x\left[\cos(\ln x) + \sin(\ln x)\right] + C$
b)	$\int \frac{1-2x}{2x^2+1}\ \mathrm{d}x$ $= \frac{1}{2}\int \frac{1}{x^2+\frac{1}{2}}\ \mathrm{d}x - \frac{1}{2}\int \frac{4x}{2x^2+1}\ \mathrm{d}x$ $= \frac{\sqrt{2}}{2}\tan^{-1}\sqrt{2}x - \frac{1}{2}\ln\left(2x^2+1\right) + c$

Q2

(a)

$$\frac{dx}{d\theta} = 3\sec\theta\tan\theta$$

$$\int_{3\sqrt{2}}^{6} \frac{3x+1}{\sqrt{x^2-9}}\, dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{9\sec\theta+1}{\sqrt{9\sec^2\theta-9}}(3\sec\theta\tan\theta)\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{9\sec\theta+1}{3\tan\theta}(3\sec\theta\tan\theta)\, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} 9\sec^2\theta + \sec\theta\, d\theta$$

$$= \left[9\tan\theta + \ln\left|\sec\theta+\tan\theta\right|\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= 9\tan\frac{\pi}{3} + \ln\left|\sec\frac{\pi}{3}+\tan\frac{\pi}{3}\right| - \left(9\tan\frac{\pi}{4} + \ln\left|\sec\frac{\pi}{4}+\tan\frac{\pi}{4}\right|\right)$$

$$= 9\sqrt{3} + \ln\left|2+\sqrt{3}\right| - \left(9+\ln\left|\sqrt{2}+1\right|\right)$$

$$= 9\sqrt{3} - 9 + \ln\frac{2+\sqrt{3}}{\sqrt{2}+1}$$

(b)(i)

Consider $y = 2 \pm 2\sqrt{1-\frac{x^2}{16}}$

Required area $= \int_{-3}^{3} 2+2\sqrt{1-\frac{x^2}{16}}\,dx - 2(6)$

$= 10.753$ (3 dp)

Alternative

Consider $\frac{x^2}{16}+\frac{y^2}{4}=1 \Rightarrow y = \pm 2\sqrt{1-\frac{x^2}{16}}$

Required area $= \int_{-3}^{3} 2\sqrt{1-\frac{x^2}{16}}\,dx$ or $4\int_{0}^{3}\sqrt{1-\frac{x^2}{16}}\,dx$

$= 10.753$ (3 dp)

(ii)

When $x=3$, $y = 2+2\sqrt{1-\frac{9}{16}} = 2+\frac{1}{2}\sqrt{7}$

When $x=0$, $y=4$

Required Volume $= \frac{\sqrt{7}}{2}\pi(3^2) + \pi\int_{2+\frac{1}{2}\sqrt{7}}^{4} 16\left(1-\frac{(y-2)^2}{4}\right)dy$

$= \frac{9\sqrt{7}}{2}\pi + 16\pi\int_{2+\frac{1}{2}\sqrt{7}}^{4} 1-\frac{(y-2)^2}{4}\,dy$

$= \frac{9\sqrt{7}}{2}\pi + 16\pi\left[y-\frac{(y-2)^3}{12}\right]_{2+\frac{1}{2}\sqrt{7}}^{4}$

$= 18\pi + 16\pi\left[4-\frac{2}{3}-2-\frac{\sqrt{7}}{2}+\frac{7\sqrt{7}}{96}\right]$

$= \frac{1}{3}\left(64-7\sqrt{7}\right)\pi$

Alternative

When $x=3$, $y = 2\sqrt{1-\frac{9}{16}} = \frac{1}{2}\sqrt{7}$

When $x=0$, $y=2$

Required Volume $= \frac{\sqrt{7}}{2}\pi(3^2) + \pi\int_{\frac{1}{2}\sqrt{7}}^{2} 16\left(1-\frac{y^2}{4}\right)dy$

$= \frac{9\sqrt{7}}{2}\pi + 16\pi\int_{\frac{1}{2}\sqrt{7}}^{2} 1-\frac{y^2}{4}\,dy$

$= \frac{9\sqrt{7}}{2}\pi + 16\pi\left[y-\frac{y^3}{12}\right]_{\frac{1}{2}\sqrt{7}}^{2}$

$= \frac{9\sqrt{7}}{2}\pi + \frac{1}{6}\left[128-41\sqrt{7}\right]\pi$

$= \frac{1}{3}\left(64-7\sqrt{7}\right)\pi$

Q3

4(i)

$x = t^2, \qquad y = t - t^3.$

$\frac{\mathrm{d}x}{\mathrm{d}t} = 2t, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = 1 - 3t^2$

$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t} = \frac{1-3t^2}{2t}$

At P, $x = p^2, \quad y = p - p^3, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1-3p^2}{2p}$

Equation of tangent at P:

$\frac{y-(p-p^3)}{x-p^2} = \frac{1-3p^2}{2p}$

$\Rightarrow 2py - 2p(p-p^3) = (x-p^2)(1-3p^2)$

$\Rightarrow 2py - 2p^2 + 2p^4 = x(1-3p^2) - p^2 + 3p^4$

$\Rightarrow 2py = x(1-3p^2) + p^2 + p^4$ (shown) -------(1)

(ii)

At A, substitute $x = 6, y = 5$ into eqn (1)

$2p(5) = 6(1-3p^2) + p^2 + p^4$

$10p = 6 - 18p^2 + p^2 + p^4$

$p^4 - 17p^2 - 10p + 6 = 0$

From GC, $p = 4.35$ (rejected) or $p = -3.7261$ or $p = -1$ or $p = 0.370$ (rejected)

Hence coordinates of P: (1,0) and (13.9, 48.0)

(iii)

Required area $= -\int_0^1 y \ \mathrm{d}x$

$= -\int_0^{-1} (t - t^3)(2t) \ \mathrm{d}t$

$= 0.267 \text{ unit}^2$

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