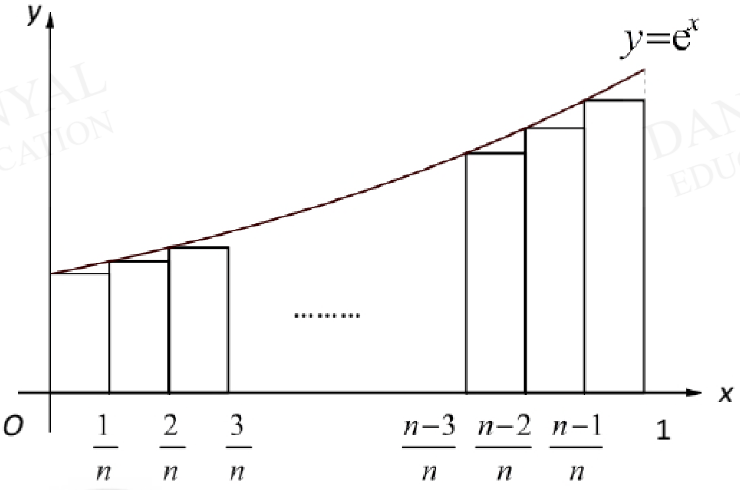
A Level H2 Math

Integration Test 11

Q1

The graph of $y = \mathrm{e}^x$, for $0 \le x \le 1$, is shown in the diagram below. Rectangles, each of width $\frac{1}{n}$ where n is an integer, are drawn under the curve.



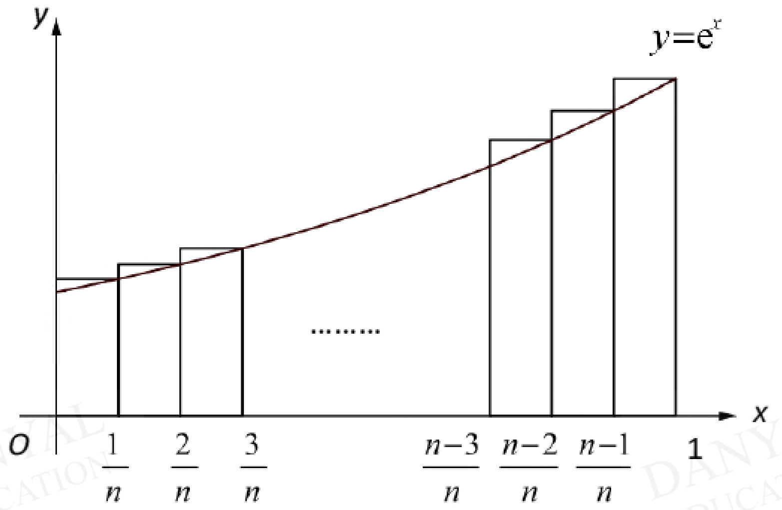
(i) Show that the total area of all the n rectangles, A_n, is $\dfrac{c}{n\left(\mathrm{e}^{\frac{1}{n}}-1\right)}$, where c is an exact constant to be found. [3]

(ii) By considering the Maclaurin Series for $\mathrm{e}^x - 1$, or otherwise, find the value of $\lim\limits_{x\to 0}\frac{1}{x}\left(\mathrm{e}^x-1\right)$. [3]

(iii) Hence, without using integration, find the exact value of $\lim\limits_{n\to\infty} A_n$. [2]

(iv) Give a geometrical interpretation of the value you found in part **(iii)**, and verify your answer in part **(iii)** using integration. [2]

Another set of n rectangles are drawn, as shown in the diagram below.



The total area of all the n rectangles in the second diagram is denoted by B_n . By considering the concavity of the graph of $y=e^x$, or otherwise, show that

$$\frac{A_n+B_n}{2}>\int_0^1 e^x dx$$

for any positive integer n. [2]

Q2

The curve C has equation $y=\sin 2x+2\cos x$, $0\le x\le 2\pi$.

(i) Using an algebraic method, find the exact x-coordinates of the stationary points. [You do not need to determine the nature of the stationary points.] [3]

(ii) Sketch the curve C, indicating clearly the coordinates of the turning points and the intersection with the axes. [1]

(iii) Find the area bounded by the curve C and the line $y=\frac{1}{\pi}x$. [3]

Q3

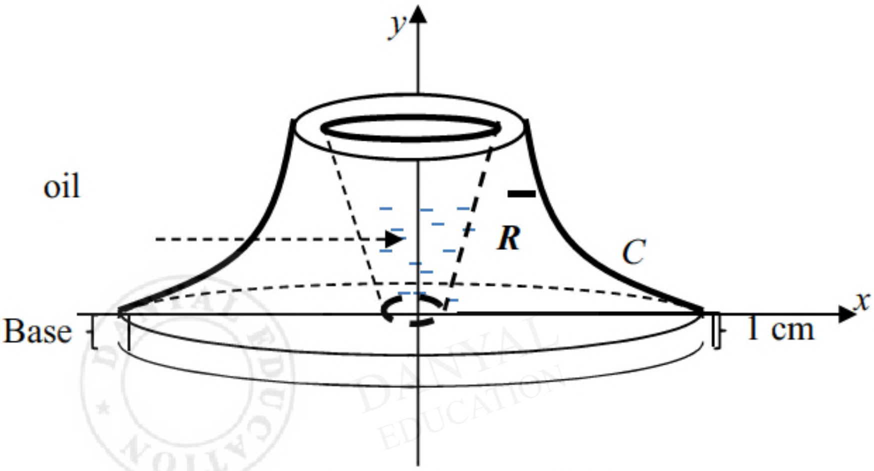
A curve C has parametric equation defined by

$$x = 4\sec t \text{ and } y = 8(1 - \tan t), \text{ where } -\tfrac{1}{4}\pi \le t \le \tfrac{1}{4}\pi .$$

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t and hence show that the equation of tangent at the point $t = -\frac{1}{6}\pi$ is $y = 4x + 8(1-\sqrt{3})$. [3]

(ii) Find the Cartesian equation of C. [2]

R is the region bounded by C, the tangent in **(i)**, the normal to C at $t = 0$ and the x-axis. Part of an oil burner is formed by rotating R completely about the y-axis as shown in the diagram below (not drawn to scale). The base of the burner is a solid cylinder of thickness 1 cm. [You may assume each unit along the x and y axis to be 1 cm]



(iii) Find the volume of the material required to make the burner. [6]

Integration Test 11

Answers

Q1

(i)

$$A_n = \frac{1}{n}\left(e^0 + e^{\frac{1}{n}} + e^{\frac{2}{n}} + e^{\frac{3}{n}} + \cdots + e^{\frac{n-2}{n}} + e^{\frac{n-1}{n}} \right)$$

$$= \frac{1}{n} \cdot \frac{e^0\left(1-\left(e^{\frac{1}{n}}\right)^n\right)}{1-e^{\frac{1}{n}}}$$

$$= \frac{1}{n} \cdot \frac{1-e}{1-e^{\frac{1}{n}}} = \frac{e-1}{n\left(e^{\frac{1}{n}}-1\right)}$$

$$\therefore c = e-1$$

(ii)

$$e^x - 1 = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right) - 1 = x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

$$\lim_{x\to 0}\frac{1}{x}\left(e^x - 1\right) = \lim_{x\to 0}\left[\frac{1}{x}\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right)\right]$$

$$= \lim_{x\to 0}\left(1 + \frac{x}{2!} + \frac{x^2}{3!} + \ldots\right)$$

$$= 1$$

(iii)

$$\lim_{n\to\infty}\frac{e-1}{n\left(e^{\frac{1}{n}}-1\right)} = \lim_{x\to 0}\frac{e-1}{\frac{1}{x}\left(e^x - 1\right)}$$

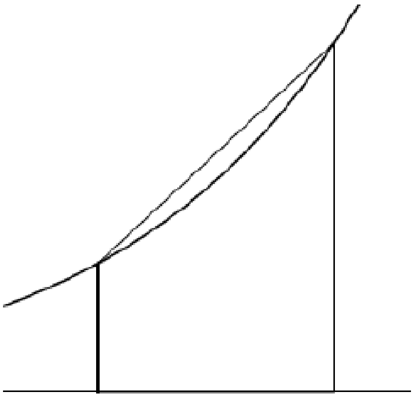
$$= e-1$$

(iv)

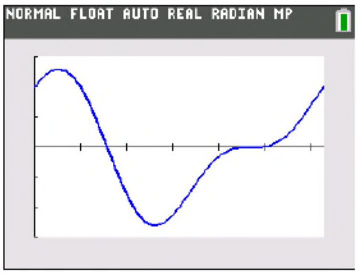
e – 1 is the exact area under the graph of $y = e^x$ from $x = 0$ to $x = 1$.

area = $\int_0^1 e^x \mathrm{d}x = e - 1$.

Since the graph of $y = e^x$ is concave upwards, and $\frac{A_n + B_n}{2}$ is the sum of the area of n trapeziums each of width $\frac{1}{n}$, the area of all trapeziums will be greater than the exact area under the graph, which is $\int_0^1 e^x \mathrm{d}x$.



Q2

(i)	$y = \sin 2x + 2\cos x$ $\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos 2x - 2\sin x$ For stationary points, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ $2\left[1 - 2\sin^2 x\right] - 2\sin x = 0$ $\Rightarrow 2\sin^2 x + \sin x - 1 = 0$ $\Rightarrow (2\sin x - 1)(\sin x + 1) = 0$ $\Rightarrow \sin x = 0.5 \quad \text{or} \quad \sin x = -1$ $\Rightarrow x = \frac{\pi}{6},\ x = \frac{5\pi}{6} \quad \text{or} \quad x = \frac{3\pi}{2}$	Differentiate and set $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ to find stationary points. As algebraic method is required, clear working of how the roots are arrived is expected, with usage of trigonometric identities along the way.
(ii)		
(iii)	From GC, the line $y = \frac{1}{\pi}x$ intersects the curve C at $x = 1.4544031$ Required area $= \int_{1..4544031}^{\frac{5\pi}{6}} \left[\frac{1}{\pi}x - (\sin 2x + 2\cos x)\right] \mathrm{d}x$ $= 2.48$ (to 3 sig figs)	In order to find the area bounded by two curves, it is most important to find where the two curves intersect first, which can be done quickly using GC.

Marker's comments

Common mistakes:

1. In (i), it is unnecessary to convert $y = \sin 2x + 2\cos x = 2\sin x\cos x + 2\cos x$ because it makes the differentiation more complicated. Students should have an awareness of the approach required by the question before manipulating the given information.

   An even more serious problem was that many students were unable to solve $2\cos 2x - 2\sin x = 0$ because identities were not used to convert it into a quadratic equation. Many were also unable to solve $\sin x = 0.5$ (forgetting about the roots in other quadrants), or $\sin x = -1$ (rejecting it immediately without finding the basic angle).

2. Students were unable to identify the correct region, which resulted in them not finding the intersection between the two curves. Also, many students did not apply that the result $\int \mathrm{f}(x) - \mathrm{g}(x)\mathrm{d}x$ to find the area of the region bounded by two curves directly, and instead tried to find the area of the individual pieces which more often than not led to errors.

Q3

(i) $x=4\sec t$ and $y=8(1-\tan t)$ $\frac{\mathrm{d}x}{\mathrm{d}t}=4\sec t\tan t$, $\frac{\mathrm{d}y}{\mathrm{d}t}=-8\sec^2 t$ $\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2}{\sin t}$ At $t=-\frac{\pi}{6}$, gradient of tangent = 4 , $x=\frac{8}{3}\sqrt{3}$ and $y=8\left(1+\frac{\sqrt{3}}{3}\right)$ Equation of tangent is $y-8\left(1+\frac{\sqrt{3}}{3}\right)=4\left(x-\frac{8\sqrt{3}}{3}\right)$ $y=4x+8\left(1-\sqrt{3}\right)$ (Shown)	Students need to know that : $\sin(-x)=-x$ $\cos(-x)=x$ $\tan(-x)=-x$

(ii) $x = 4\sec t \Rightarrow \sec^2 t = \dfrac{x^2}{16}$

$$y = 8(1-\tan t) \Rightarrow \tan^2 t = \left(1-\frac{y}{8}\right)^2$$

Since $1+\tan^2 x = \sec^2 x$,

$$1+\left(1-\frac{y}{8}\right)^2 = \frac{x^2}{16}$$

$$\frac{x^2}{16} - \frac{(y-8)^2}{64} = 1$$

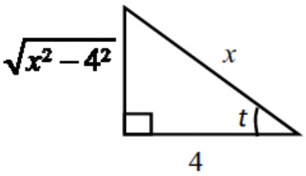
where $4 \le x \le 4\sqrt{2}$ and $0 \le y \le 16$ $\left(\because -\dfrac{\pi}{4} \le t \le \dfrac{\pi}{4}\right)$

Alternative method:

$\sec t = \dfrac{x}{4} \Rightarrow \cos t = \dfrac{4}{x}$

$y = 8(1-\tan t)$

$$y = 8\left(1 \pm \frac{\sqrt{x^2-16}}{4}\right)$$



(Note that $-\dfrac{\pi}{4} \le t \le \dfrac{\pi}{4} \Rightarrow \tan t = \dfrac{\sqrt{x^2-16}}{4}$ or $-\dfrac{\sqrt{x^2-16}}{4}$)

Note that $y = 8\left(1-\tan\left(\cos^{-1}\dfrac{4}{x}\right)\right)$ is not in its simplest form

When C intersects x-axis, $y = 0$,

$$\frac{x^2}{16} - \frac{(0-8)^2}{64} = 1 \Rightarrow x^2 = 32$$

$$x = 4\sqrt{2} \quad (\because \text{radius} > 0)$$

Volume of cylindrical base $= \pi\left(\sqrt{2}(4)\right)^2(1) = 32\pi$

Method 1

Volume of the solid that made the burner

$$= \frac{\pi}{4}\int_0^8 64+(y-8)^2 \, dy - \frac{\pi}{16}\int_0^8 \left(y-8\left(1-\sqrt{3}\right)\right)^2 dy + 32\pi$$

$\approx 475.718 = 476$ units^3 (using GC)

Method 2

Volume of solid that made the burner

$$= \pi\int_{\frac{\pi}{4}}^{0} (4\sec t)^2\left(-8\sec^2 t\right) dt - \frac{\pi}{16}\int_0^8 \left(y-8\left(1-\sqrt{3}\right)\right)^2 dy + 32\pi \approx 476$$

Method 3

Volume of solid that made the burner

$$= \frac{\pi}{4}\int_0^8 64+(y-8)^2\ dy+32\pi - \left[\underbrace{\frac{1}{3}\pi\left(2\sqrt{3}\right)^2\left(8+8\left(\sqrt{3}-1\right)\right)}_{\text{Volume of larger cone}} - \underbrace{\frac{1}{3}\pi\left(2\sqrt{3}-2\right)^2\left(8\left(\sqrt{3}-1\right)\right)}_{\text{Vol. of smaller cone}}\right]$$

≈ 476 units^3

Students who use Method 3 need to realise that when finding height of the two cones, for example, the height of the larger cone, they should not be using $8+8\left(1-\sqrt{3}\right)$ since $8\left(1-\sqrt{3}\right)$ is a negative y-intercept.

Marker’s comments

(i) Generally well done.

Common errors is not knowing when to have negative sign when evaluating:

$$\tan\left(-\frac{\pi}{6}\right) \text{ and } \sec\left(-\frac{\pi}{6}\right)$$

(ii) Many students have forgotten the meaning of Cartesian equation, ended up with an equation that contains the parameter t which is wrong.

Quite a number of students leave answer as $y=8\left(1-\tan\left(\cos^{-1}\frac{4}{x}\right)\right)$ but this is not in the simplest form.

A serious mistake made by some students is to attempt to integrate $\frac{dy}{dx}=-\frac{2}{\sin t}$ but without realising that they cannot integrate $-\frac{2}{\sin t}$ with respect to x.

Last part

This part is very badly done.

Many either leave it blank or made a lot of careless/algebraic manipulation mistakes when trying to find x^2 in terms of y.