A Level H2 Math

Integration Test 10

Q1

A curve C has parametric equations

$$x = 1 + e^{t} + e^{-t}, \quad 2y = e^{t} - e^{-t}, \quad t \in \mathbb{R}.$$

(i) Show that the Cartesian equation of C is $\frac{(x-1)^2}{2^2} - y^2 = 1$. [2]

(ii) Sketch C, showing clearly the equations of any asymptotes and coordinates of the centre and the point(s) where the curve cuts the x-axis. [3]

(iii) Find the exact area of the region bounded by C and the line $x = 1 + e + e^{-1}$. [4]

(iv) Find the volume of the solid of revolution when the region bounded by C and the lines $x = 3$ and $y = 4$ is rotated completely about the y-axis. [2]

Q2

Suppose a point P on the rim of a wheel of radius r is initially at the point O. As the wheel roll along the x-axis without slippage, the locus of P, known as a *cycloid*, has parametric equations given by

$$x = r(\theta - \sin\theta),\ y = r(1 - \cos\theta),\ \theta \geq 0.$$

(i) Sketch the locus of P for $0 \leq \theta \leq 4\pi$. [2]

(ii) Show that $\frac{dy}{dx} = \cot\frac{\theta}{2}$. [3]

(iii) Show that the curve is a solution to the differential equation $\left(\frac{dy}{dx}\right)^2 = \frac{2r}{y} - 1$. [3]

(iv) Find the exact area bounded by the locus of P and the x-axis for $0 \leq x \leq 2\pi r$. [4]

Q3

A curve C has parametric equations

$$x=\sqrt{2}\cos\frac{t}{2},\quad y=\sqrt{2}\sin t,\quad \text{for } -2\pi\le t\le 2\pi.$$

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ and verify that curve C has a stationary point at P with parameter $\frac{\pi}{2}$. Hence find the equation of the normal to the curve at point P. [3]

(ii) Sketch C, indicating clearly all turning points and axial intercepts in exact form. [4]

(iii) Find the exact area bounded by the curve C. (You may first consider the area bounded by the curve C and the positive x-axis in the first quadrant.) [6]

Integration Test 10

Answers

Q1

(i)

$(x-1)^2 = (e^t + e^{-t})^2 = e^{2t} + 2 + e^{-2t}$

$(2y)^2 = (e^t - e^{-t})^2 = e^{2t} - 2 + e^{-2t}$

Hence $(x-1)^2 - (2y)^2 = 4$

$$\frac{(x-1)^2}{2^2} - y^2 = 1$$

Alternative solution by students:

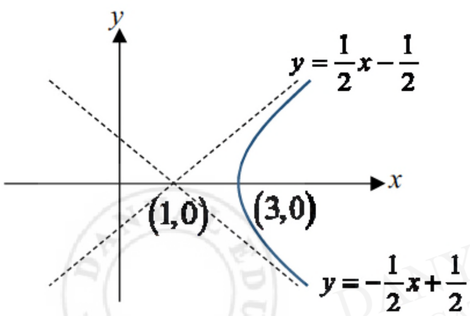
$(x-1) + 2y = 2e^t$ ___(1)

$(x-1) - 2y = 2e^{-t}$ ___(2)

(1)×(2):

$(x-1)^2 - (2y)^2 = 4e^t e^{-t} = 4$

(ii)



(iii)

When $x = 3$, $3 = 1 + e^t + e^{-t}$

$e^t + e^{-t} = 2$

$t = 0$

When $x = 1 + e + e^{-1}$, $t = \pm 1$ ($t = 1$: $y > 0$, $t = -1$: $y < 0$)

$x = 1 + e^t + e^{-t}$

$\frac{dx}{dt} = e^t - e^{-t}$

By symmetry

Note that the integral $\int y\,dx$ refers to the area either above the x-axis (if $y > 0$) or below the x-axis (if $y < 0$)

Area of required region $= 2\int_3^{1+e+e^{-1}} y\,dx$

$$= 2\int_0^1 \frac{e^t - e^{-t}}{2}(e^t - e^{-t})\,dt$$

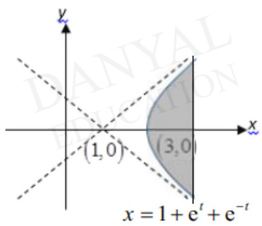
$$= \int_0^1 (e^t - e^{-t})^2\,dt$$

$$= \int_0^1 (e^{2t} - 2 + e^{-2t})\,dt$$

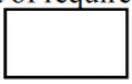
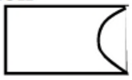
$$= \left[\frac{1}{2}e^{2t} - 2t - \frac{1}{2}e^{-2t}\right]_0^1$$

$$= \left[\frac{1}{2}e^2 - 2 - \frac{1}{2}e^{-2}\right] - 0$$

$$= \frac{1}{2}(e^2 - e^{-2}) - 2$$



Alternatively (more tedious):
Area of required region

=  − 

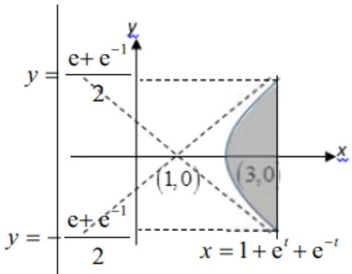
$$=\left(1+e+e^{-1}\right)\left(\frac{e-e^{-1}}{2}-\left(-\frac{e-e^{-1}}{2}\right)\right)-\int_{-\frac{e-e^{-1}}{2}}^{\frac{e-e^{-1}}{2}} x\ \mathrm{d}y$$

$$=\left(1+e+e^{-1}\right)\left(e-e^{-1}\right)-2\int_{0}^{\frac{e-e^{-1}}{2}} x\ \mathrm{d}y$$

$$=\left(1+e+e^{-1}\right)\left(e-e^{-1}\right)-2\int_{0}^{1}\left(1+e^{t}+e^{-t}\right)\frac{e^{t}+e^{-t}}{2}\,\mathrm{d}t$$

$$\vdots$$

$$=\frac{1}{2}\left(e^{2}-e^{-2}\right)-2$$



(iv)

$$\frac{(x-1)^2}{2^2}-y^2=1$$

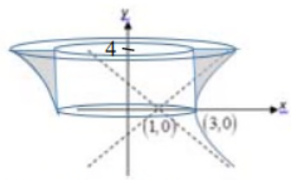
$$(x-1)^2=2^2\left(1+y^2\right)$$

$$x=1+2\sqrt{1+y^2} \quad \text{since } x>1$$

$$\text{Volume } =\pi\int_0^4 x^2\mathrm{d}y-\pi\left(3^2\right)(4)$$

$$=\pi\int_0^4\left(1+2\sqrt{1+y^2}\right)^2\mathrm{d}y-36\pi$$

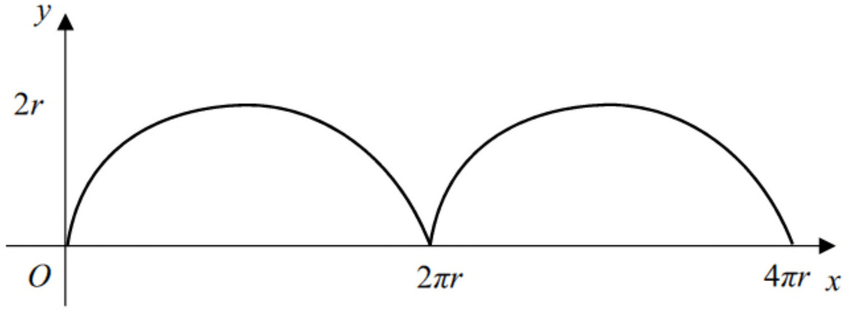
$$=335 \text{ units}^3 \quad (3 \text{ s.f.})$$



Note that finding volume of revolution when the curve is defined parametrically is not in syllabus. Students can use the parametric equations to find volume but are not expected to do so. You should just use the Cartesian equation.

Q2

(i)



(ii)

$$\frac{dx}{d\theta} = r(1-\cos\theta),\ \frac{dy}{d\theta} = r\sin\theta$$

$$\frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$$

$$= \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}$$

Use do formula

Use trigo identity

$$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2}$$

(iii)

$$\left(\frac{dy}{dx}\right)^2 = \cot^2\frac{\theta}{2} = \operatorname{cosec}^2\frac{\theta}{2} - 1$$

$$= \frac{1}{\sin^2\frac{\theta}{2}} - 1$$

Use double angle formula

$$= \frac{2}{1-\cos\theta} - 1 = \frac{2r}{r(1-\cos\theta)} - 1$$

Thus $\left(\frac{dy}{dx}\right)^2 = \frac{2r}{y} - 1$

(iv)

$$Area = \int_0^{2\pi r} y\,dx$$

$$= \int_0^{2\pi} r(1-\cos\theta)\cdot r(1-\cos\theta)\,d\theta$$

$dx = \frac{dx}{d\theta}d\theta = r(1-\cos\theta)d\theta$

$$= r^2\int_0^{2\pi}(1-2\cos\theta+\cos^2\theta)\,d\theta$$

Use double angle formula

$$= r^2\int_0^{2\pi}\left(\frac{3}{2} - 2\cos\theta + \frac{1}{2}\cos 2\theta\right)d\theta$$

$$= r^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= 3\pi r^2$$

Q3

(i) $x=\sqrt{2}\cos\frac{t}{2} \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t}=-\frac{\sqrt{2}}{2}\sin\frac{t}{2}$

$y=\sqrt{2}\sin t \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t}=\sqrt{2}\cos t$

$\therefore \frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2\cos t}{\sin\frac{t}{2}}$

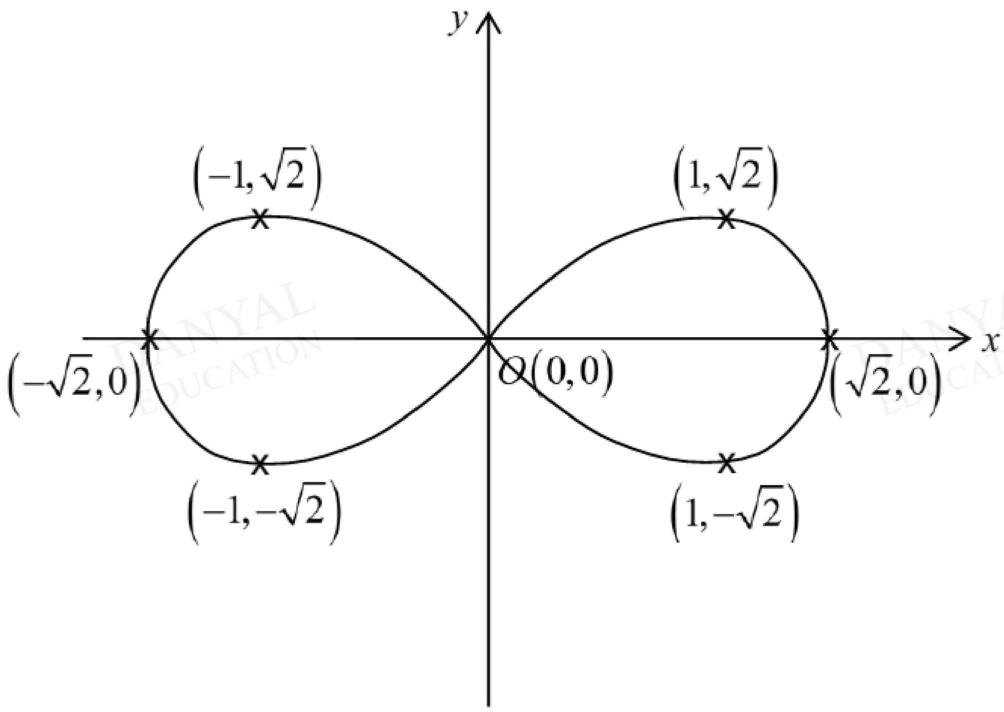
At $t=\frac{\pi}{2}$,

$\frac{\mathrm{d}y}{\mathrm{d}x}=-\frac{2\cos\frac{\pi}{2}}{\sin\frac{\pi}{4}}=0$ (verified)

When $t=\frac{\pi}{2}$, $x=\sqrt{2}\cos\left(\frac{\pi}{4}\right)=1$

Equation of normal: $x=1$

(ii)



(iii)

$$\text{Area} = 4\int_0^{\sqrt{2}} y \, \mathrm{d}x$$

$$= 4\int_\pi^0 \sqrt{2}\sin t \cdot \left(-\frac{\sqrt{2}}{2}\sin\frac{t}{2}\right) \mathrm{d}t$$

$$= 4\int_0^\pi \sin t \cdot \sin\frac{t}{2} \, \mathrm{d}t$$

$$= 8\int_0^\pi \sin^2\frac{t}{2}\cos\frac{t}{2} \, \mathrm{d}t$$

$$= 8\left[\frac{2}{3}\sin^3\frac{t}{2}\right]_0^\pi$$

$$= \frac{16}{3} \text{ units}^2$$

Alternative Method

$$\text{Area} = 4\int_0^{\sqrt{2}} y \, \mathrm{d}x$$

$$= 4\int_\pi^0 \sqrt{2}\sin t \cdot \left(-\frac{\sqrt{2}}{2}\sin\frac{t}{2}\right) \mathrm{d}t$$

$$= 4\int_0^\pi \sin t \cdot \sin\frac{t}{2} \, \mathrm{d}t$$

$$= -2\int_0^\pi \cos\frac{3t}{2} - \cos\frac{t}{2} \, \mathrm{d}t$$

$$= -2\left[\frac{2}{3}\sin\frac{3t}{2} - 2\sin\frac{t}{2}\right]_0^\pi$$

$$= \frac{16}{3} \text{ units}^2$$