A Level H2 Math

Integration Test 1

Q1

Solve the inequality $\frac{1}{x+a} \leq \frac{2a}{x^2-a^2}$, leaving your answer in terms of a, where a is a positive real number. [3]

Hence or otherwise, find $\int_{2a}^{4a} \left| \frac{1}{x+a} - \frac{2a}{x^2-a^2} \right| \mathrm{d}x$ exactly. [4]

Q2

(i) Find $\int \frac{x}{\left(1+x^2\right)^2} \,\mathrm{d}x$. [2]

(ii) By using the substitution $x = \tan\theta$, show that

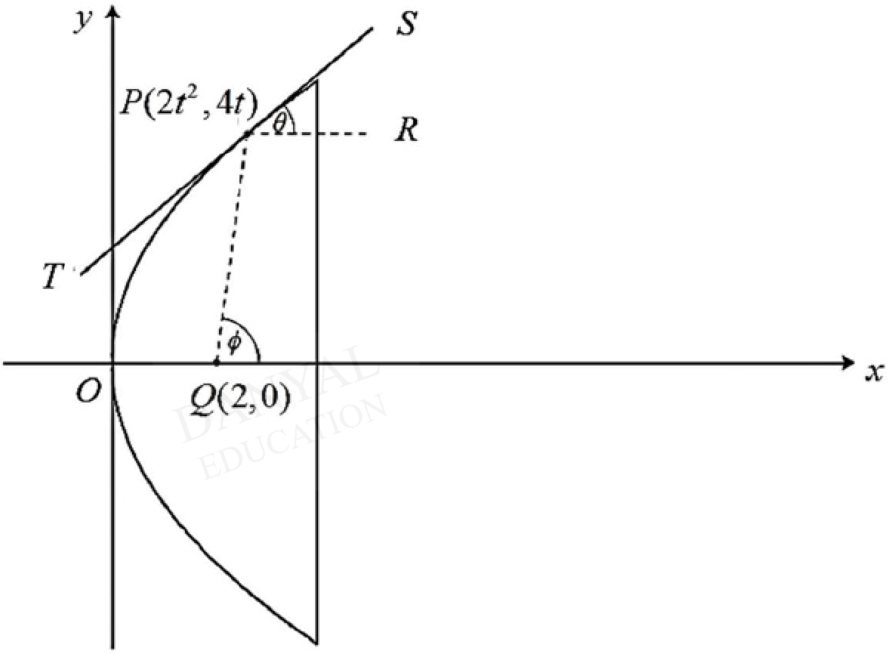
$$\int \frac{1}{\left(1+x^2\right)^2} \,\mathrm{d}x = k\left(\frac{x}{1+x^2} + \tan^{-1} x\right) + c,$$

where c is an arbitrary constant, and k is a constant to be determined. [5]

(iii) Hence find $\int \frac{x^2}{\left(1+x^2\right)^2} \,\mathrm{d}x$. [3]

(iv) Using all of the above, find $\int \frac{x^2+2x+5}{\left(1+x^2\right)^2} \,\mathrm{d}x$, simplifying your answer. [2]

Q3



The figure above shows a cross-section of a searchlight whose inner reflective surface is modelled, in suitable units, by the curve

$$x = 2t^2, \quad y = 4t, \quad -\sqrt{2} \le t \le \sqrt{2}.$$

The inner reflective surface of the searchlight has the shape produced by rotating the curve about the x-axis.

(i) Show that the curve has cartesian equation $y^2 = 8x$, and find the volume of revolution of the curve, giving your answer as a multiple of π. [3]

$P(2t^2, 4t)$ is a point on the curve with parameter t. TS is the tangent to the curve at P, and PR is the line through P parallel to the x-axis. Q is the point $(2, 0)$. The angles that PS and QP make with the positive x-direction are θ and ϕ respectively.

(ii) By considering the gradient of the tangent TS, show that $\cot\theta = t$. [2]

(iii) Find the gradient of the line QP in terms of t. Hence show that $\phi = 2\theta$, and show that angle TPQ is equal to θ. [5]

A lamp bulb is placed at Q.

(iv) Use your answer to part **(iii)** to describe the direction of the reflected light from the bulb. [1]

(v) Find a cartesian equation of the locus of the mid-point M on PQ as t varies. [2]

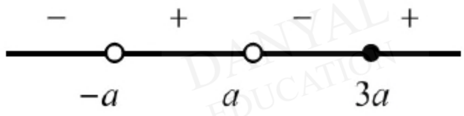
Answers

Integration Test 1

Q1

$$\frac{1}{x+a} \le \frac{2a}{x^2-a^2} \Rightarrow \frac{1}{x+a} - \frac{2a}{x^2-a^2} \le 0$$

$$\Rightarrow \frac{x-3a}{(x+a)(x-a)} \le 0$$



$$\therefore x < -a \text{ or } a < x \le 3a$$

$$\int \frac{1}{x+a} - \frac{2a}{x^2-a^2}\,\mathrm{d}x = \ln(x+a) - \ln\left(\frac{x-a}{x+a}\right) = \ln\frac{(x+a)^2}{x-a}$$

OR

$$\int \frac{x-3a}{(x+a)(x-a)}\,\mathrm{d}x = \int \frac{2}{x+a} - \frac{1}{x-a}\,\mathrm{d}x = \ln\frac{(x+a)^2}{x-a}$$

$$\int_{2a}^{4a} \left| \frac{1}{x+a} - \frac{2a}{x^2-a^2} \right| \mathrm{d}x$$

$$= \int_{2a}^{3a} -\left(\frac{1}{x+a} - \frac{2a}{x^2-a^2}\right)\mathrm{d}x + \int_{3a}^{4a} \frac{1}{x+a} - \frac{2a}{x^2-a^2}\,\mathrm{d}x$$

Many students still are unfamiliar with the basics of solving inequalities and lack the basic skills of factorisation:

(1) Do not know how to find the lowest common multiple of the denominators. Many gave $(x^2-a^2)(x+a)$ as the denominator instead of $(x-a)(x+a)$. Those who did so made a mess out of the numerator and could not factorise the numerator properly.

(2) Many did not even know how to factorise x^2-a^2.

(3) Many insisted on removing the denominator and change the inequality to an inequality involving polynomial only. However they could not do it properly and made a mess out of the polynomial and could not factorise.

(4) For those using graphical method, they attempted to draw the graph of $y = \frac{1}{x+a} - \frac{2a}{x^2-a^2}$ and did not do it properly. They most likely just copied the graph from G.C. without drawing the horizontal asymptote.

(5) Whether by using the sign test with number line or using the graphical method, students still could not obtain the answer correctly, giving the wrong range of values of x.

$$\int_{2a}^{3a} -\left(\frac{1}{x+a}-\frac{2a}{x^2-a^2}\right)\mathrm{d}x+\int_{3a}^{4a}\frac{1}{x+a}-\frac{2a}{x^2-a^2}\,\mathrm{d}x$$

$$=-\left[\ln\frac{(x+a)^2}{x-a}\right]_{2a}^{3a}+\left[\ln\frac{(x+a)^2}{x-a}\right]_{3a}^{4a}$$

$$=-\left(\ln\frac{16a^2}{2a}-\ln\frac{9a^2}{a}\right)+\left(\ln\frac{25a^2}{3a}-\ln\frac{16a^2}{2a}\right)$$

$$=-\ln\frac{8}{9}+\ln\frac{25}{24}=\ln\left(\frac{25}{24}\times\frac{9}{8}\right)=\ln\frac{75}{64}$$

Even some of the values for the x-intercept and vertical asymptotes, $x=-a, x=a, x=3a$ were incorrect particularly, $x=3a$. Even for those who did almost everything correct included $x=-a, x=a$ as part of the answer.

For integration, very few students use Partial Fractions but used the formula in MF26 to integrate directly and most people applied the formula correctly. Most people could carry out the integration properly but could not obtain the final simplified answer $\ln\frac{75}{64}$. There were quite a number of students who apply the formula $\int\frac{1}{x^2-a^2}\,dx=\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)+c$ to $\int\left|\frac{1}{x^2-a^2}\right|dx=\left|\frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right)\right|+c$.

Some even carried forward the polynomial obtained in the earlier portion for the question on inequality to replace fractions $\frac{1}{x+a}-\frac{2a}{x^2-a^2}$ as the integrand.

Q2

(i)

$$\int \frac{x}{(1+x^2)^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{2x}{(1+x^2)^2}\,\mathrm{d}x = -\frac{1}{2(1+x^2)} + c$$

This is a simple question. No one should be getting this wrong.

(ii)

$$x = \tan\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = \sec^2\theta$$

$$\int \frac{1}{(1+x^2)^2}\,\mathrm{d}x = \int \frac{1}{\left(1+\tan^2\theta\right)^2}\sec^2\theta\,\mathrm{d}\theta$$

$$= \int \frac{1}{\sec^2\theta}\,\mathrm{d}\theta = \int \cos^2\theta\,\mathrm{d}\theta$$

$$= \int \frac{\cos 2\theta + 1}{2}\,\mathrm{d}\theta = \frac{1}{2}\left(\frac{\sin 2\theta}{2} + \theta\right)$$

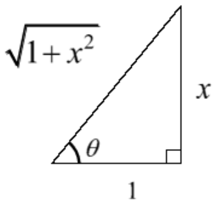
$$= \frac{1}{2}\left(\sin\theta\cos\theta + \theta\right) + c$$

$$= \frac{1}{2}\left(\frac{x}{1+x^2} + \tan^{-1}x\right) + c$$

$x = \tan\theta$

$\sin\theta = \frac{x}{\sqrt{1+x^2}}$

$\cos\theta = \frac{1}{\sqrt{1+x^2}}$



(ii) was done better than (i) in general.
A significant minority did not know that $1+\tan^2\theta = \sec^2\theta$ though, and either got stuck or used very long methods to get to a integrand they could work with.
As this is a **show** question, students have to present the way they substitute the variable x back into the integral clearly, either using the triangle or with identities.
This was quite poorly done though a lot of leeway was given in the awarding of marks.

(iii)

$$\int \frac{x^2}{(1+x^2)^2}\,\mathrm{d}x = \int \frac{x^2+1-1}{(1+x^2)^2}\,\mathrm{d}x = \int \frac{1}{1+x^2} - \frac{1}{\left(1+x^2\right)^2}\,\mathrm{d}x$$

$$= \tan^{-1}x - \frac{1}{2}\left(\frac{x}{1+x^2} + \tan^{-1}x\right) + c$$

$$= \frac{1}{2}\left(\tan^{-1}x - \frac{x}{1+x^2}\right) + c$$

There were many different methods available here, the splitting (shown on the left).
Other easy methods include:
(1) using the substitution provided in (ii).
(2) by parts with parts $\frac{x}{\left(1+x^2\right)^2}$ and x and using (i).
A long method uses the parts $\frac{1}{\left(1+x^2\right)^2}$ and x^2.
Many careless mistakes surfaced in this part (although they were prevalent throughout the question as well), such as confusing $\frac{1}{\left(1+x^2\right)^2}$ with $\frac{1}{1+x^2}$ or $\frac{1}{\left(1+x\right)^2}$.

(iv)

$$\int \frac{x^2+2x+5}{(1+x^2)^2}\,\mathrm{d}x = \int \frac{x^2}{(1+x^2)^2} + \frac{2x}{(1+x^2)^2} + \frac{5}{(1+x^2)^2}\,\mathrm{d}x$$

$$= \frac{1}{2}\left(\tan^{-1}x - \frac{x}{1+x^2}\right) + 2\left(-\frac{1}{2(1+x^2)}\right) + 5\left(\frac{1}{2}\left(\frac{x}{1+x^2} + \tan^{-1}x\right)\right) + c$$

$$= 3\tan^{-1}x + \frac{2x-1}{1+x^2} + c$$

This was generally well done, as students could use (i)-(iii). Working mark was given even if their integrals were wrong, as long as they were based on their answers in the earlier part.
The simplification of the answer was not done by a significant minority.

Q3

(i)

$$y^2 = (4t)^2 = 16t^2$$
$$= 8(2t^2)$$
$$= 8x \quad \text{(shown)}$$

$$\text{Volume} = \pi \int_0^4 8x \, dx$$
$$= \pi \left[4x^2 \right]_0^4$$
$$= 64\pi$$

(ii)

$$\frac{dx}{dt} = 4t, \quad \frac{dy}{dt} = 4$$
$$\frac{dy}{dx} = \frac{1}{t}$$

Gradient of tangent $TS = \tan\theta$

$$\therefore \tan\theta = \frac{1}{t}$$
$$\cot\theta = t \quad \text{(shown}$$

(iii)

$$\text{Gradient of line } QP = \frac{4t-0}{2t^2-2}$$
$$= \frac{2t}{t^2-1}$$
$$= \frac{2/\tan\theta}{1/\tan^2\theta - 1}$$
$$= \frac{2\tan\theta}{1-\tan^2\theta}$$
$$= \tan 2\theta$$

$\tan\phi = \tan 2\theta \Rightarrow \phi = 2\theta$ (shown)

$\angle QPR = 180° - \phi$ (interior angles)

$= 180° - 2\theta$ (by earlier results)

$\angle TPQ + (180° - 2\theta) + \theta = 180°$

$\therefore \angle TPQ = \theta$ (shown)

(iv)

The reflected light from the bulb produces a horizontal beam of light/ produces a beam of line parallel to x-axis

(v)

$$\text{Midpoint } M = \left(\frac{2+2t^2}{2}, \frac{4t+0}{2} \right)$$
$$= \left(1+t^2, \ 2t\right)$$

$$\begin{cases} x = 1+t^2 \\ y = 2t \Rightarrow t = \dfrac{y}{2} \end{cases}$$

Locus of midpoint M is:

$$x = 1 + \frac{y^2}{4}$$
$$y^2 = 4(x-1)$$