A Level H2 Math

Hypothesis Testing Test 5

Q1

The Kola Company receives a number of complaints that the volume of cola in their cans are less than the stated amount of 500 ml. A statistician decides to sample 50 cola cans to investigate the complaints. He measures the volume of cola, x ml, in each can and summarised the results as follows:

$$\sum x = 24730, \sum x^2 = 12242631.$$

(i) Find unbiased estimates of the population mean and variance correct to 2 decimal places and carry out the test at the 1% level of significance. [6]

(ii) One director in the company points out that the company should test whether the volume of cola in a can is 500 ml at the 1% significance level instead. Using the result of the test conducted in **(i)**, explain how the p-value of this test can be obtained from p-value in part **(i)** and state the corresponding conclusion. [2]

The head statistician agrees the company should test that the volume of cola in a can is 500 ml at the 1% level of significance. He intends to make a simple rule of reference for the production managers so that they will not need to keep coming back to him to conduct hypothesis tests. On his instruction sheet, he lists the following:

1. Collect a random sample of 40 cola cans and measure their volume.
2. Calculate the mean of your sample, $\bar{x}$ and the variance of your sample, $s_x^{\ 2}$.
3. Conclude that the volume of cola differs from 500 ml if the value of $\bar{x}$ lies.....

(iii) Using the above information, complete the decision rule in step 3 in terms of s_x. [4]

A party organiser has n cans of cola and $2n$ packets of grape juice. Assume now that the volume of a can of cola has mean 500 ml and variance 144 ml^2, and the volume of a packet of grape juice has mean 250 ml and variance 25 ml^2. She mixes all the cola and grape juice into a mocktail, which she pours into a 120-litre barrel. Assume that n is sufficiently large and that the volumes of the cans of cola and packets of grape juice are independent.

(iv) Show that if the party organiser wants to be at least 95% sure that the barrel will not overflow, n must satisfy the inequality $1000n + 22.9\sqrt{n} - 120,000 \leq 0$. [4]

Q2

Ryde, a leading private hire car company, announced JustRyde, a new service that promises more affordable fixed fare rides and shorter waiting times. In their advertisement, Ryde claimed that the mean waiting time, in seconds, was 240. A random sample of 50 JustRyde customers is taken and their waiting times, x seconds, is recorded. The data are summarised by

$$\Sigma(x-240)=120, \quad \Sigma(x-240)^2=11200.$$

(i) Find unbiased estimates of the population mean and variance. [2]

(ii) Test, at the 10% significance level, whether the population mean waiting time is more than 240 seconds. [5]

(iii) State, giving a valid reason, whether any assumptions about the population are needed in order for the test to be valid. [1]

(iv) Explain, in the context of the question, the meaning of 'at the 10% significance level'. [1]

(v) In another test, using the same data and also at the 10% significance level, the hypotheses are as follows:

Ho : the population mean waiting time is equal to k seconds.
H_1 : the population mean waiting time is not equal to k seconds.

Given that the null hypothesis is rejected in favour of the alternative hypothesis, find the set of possible values of k. [3]

Q3

A manufacturing process produces ball bearings with diameters with known standard deviation 0.04 cm. Under normal circumstances, the manufacturing process will produce ball bearings of mean diameter 0.5 cm.

(i) During a routine quality control check, a random sample of 25 ball bearings gives a mean of 0.51 cm. Is there evidence to believe that the manufacturing process is producing ball bearings of the stated diameter? Perform an appropriate test at 5% level of significance. State a necessary assumption for the test to be valid. [4]

An enhancement on the manufacturing process will ensure that the diameters of the ball bearings produced are less variable.

(ii) Measurements of a sample of 100 ball bearings give the following summary statistics:

$$\Sigma x = 50.6,\ \Sigma(x-0.5)^2 = 0.08345.$$

Show that the unbiased estimate of the population variance is 8.07×10^{-4}.

Is there evidence at the 5% level of significance that after the enhancement, the manufacturing process is producing oversized ball bearings? [4]

(iii) Another sample of 100 ball bearings yield the same summary statistics as the previous sample in **(ii)**. Explain, with justification, whether the combined sample will give a different conclusion to **(ii)**. [4]

Answers

Hypothesis Testing Test 5

Q1

(i)

Unbiased estimate of population mean,

$$\bar{x} = \frac{24730}{50} = 494.60$$

Unbiased estimate for population variance,

$$s^2 = \frac{1}{49}\left(12242631 - \frac{24730^2}{50}\right) = 228.02$$

Let X be the volume of beer in one beer can in ml and μ be the population mean volume of beer of the beer cans.

$H_0 : \mu = 500$

$H_1 : \mu < 500$

Under H_0, since $n = 50$ is large, by the Central Limit Theorem,

$\bar{X} \sim N\left(500, \frac{s^2}{50}\right)$ approximately.

Use a left-tailed z-test at the 1% level of significance.

Test statistic: $Z = \dfrac{\bar{X} - 500}{\frac{s}{\sqrt{50}}} \sim N(0,1)$.

Reject H_0 if p-value ≤ 0.01.

From the sample,

$p - \text{value} = 0.0057248 = 0.00572$

Since p-value $= 0.00572 \leq 0.01$, we reject H_0. There is sufficient evidence at the 1% level of significance to conclude that the volume of cola in a can is less than 500 ml.

(iii)

Let X be the volume of cola in one can in ml and μ be the population mean volume of cola of the cans.

$H_0 : \mu = 500$

$H_1 : \mu \neq 500$

Unbiased estimate of population variance,

$$s^2 = \frac{40}{39}(s_x)^2$$

Under H_0, since $n = 40$ is large, by the Central Limit Theorem,

$\bar{X} \sim N\left(500, \frac{s_x^{\,2}}{39}\right)$ approximately.

Use a two-tailed z-test at the 1% level of significance.

Test statistic: $Z = \dfrac{\bar{X} - 500}{\frac{s_x}{\sqrt{39}}} \sim N(0,1)$

Critical values: $z_{crit(1)} = -2.5758 \quad z_{crit(2)} = 2.5758$.

Reject H_0 if

$z_{cal} \leq -2.5758$ or $z_{cal} \geq 2.5758$.

Since H_0 is rejected,

$-2.5758 \le z_{cal}$ or $z_{cal} \ge 2.5758$

$-2.5758 \le \dfrac{\bar{x}-500}{\sqrt{\dfrac{s_x^2}{39}}}$ or $\dfrac{\bar{x}-500}{\sqrt{\dfrac{s_x^2}{39}}} \ge 2.5758$

$500-2.5758\sqrt{\dfrac{s_x^2}{39}} \le \bar{x}$ or $\bar{x} \ge 500+2.5758\sqrt{\dfrac{s_x^2}{39}}$

$500-0.41246s_x \le \bar{x}$ or $\bar{x} \ge 500+0.41246s_x$

$500-0.412s_x \le \bar{x}$ or $\bar{x} \ge 500+0.412s_x$

Hence the decision rule should read:
Conclude that the volume of cola differs from 500 ml if the value of $\bar{x}$ lies within this range : $500-0.412s_x \le \bar{x}$ or $\bar{x} \ge 500+0.412s_x$.

(iv)
Let X be the volume of cola in one can in ml.
since n is large, by the Central Limit Theorem,
$X_1 + X_2 + ... + X_n \sim N(500n, 144n)$ approximately.
Let Y be the volume of grape juice in one packet in ml.
since $2n$ is large, by the Central Limit Theorem,
$Y_1 + Y_2 + ... + Y_{2n} \sim N(500n, 50n)$ approximately.

$X_1 + X_2 + ... + X_n + Y_1 + Y_2 + ... + Y_{2n} \sim N(1000n, 194n)$

$P(X_1 + X_2 + ... + X_n + Y_1 + Y_2 + ... + Y_{2n} \le 120,000) \ge 0.95$

$P\left(Z \le \dfrac{120,000-1000n}{\sqrt{194n}} \right) \ge 0.95$

$\dfrac{120,000-1000n}{\sqrt{194n}} \ge 1.6449$

$120,000-1000n \ge 1.6449\sqrt{194n}$

$1000n + 22.9\sqrt{n} - 120,000 \le 0$

Q2

i

Let $y = x - 240$

unbiased estimate of population mean

$= \bar{x}$

$= \bar{y} + 240$

$= \dfrac{\sum y}{n} + 240$

$= \dfrac{120}{50} + 240$

$= 242.4$

Unbiased estimate of population variance

$= s^2$

$= \dfrac{1}{n-1}\left(\sum y^2 - \dfrac{\left(\sum y\right)^2}{n} \right)$

$= \dfrac{1}{49}\left(11200 - \dfrac{120^2}{50} \right)$

$= 222.69 = 223$ (3 s.f)

ii

Let μ be the population mean of X.

$H_0 : \mu = 240$

$H_1 : \mu > 240$

Level of significance: 10%

Test Statistic : since $n = 50$ is sufficiently large,
By Central Limit Theorem,
$\bar{X}$ is approximately normal.
When H_0 is true,

$$Z = \frac{\bar{X} - 240}{\frac{S}{\sqrt{50}}} \sim N(0,1) \text{ approximately}$$

Computation :

$\bar{x} = 242.4$

$s = \sqrt{222.69} = 14.923$

$p-\text{value} = 0.128$ (3 s.f)

Conclusion : Since p-value = 0.128 > 0.10, H_0 is not rejected at the 10% significance level. So there is insufficient evidence that the population mean waiting time is more than 240 seconds.

iii No assumption is needed. Since the sample size is large, by Central Limit Theorem, the distribution of the sample mean ($\bar{X}$) is approximately normal.

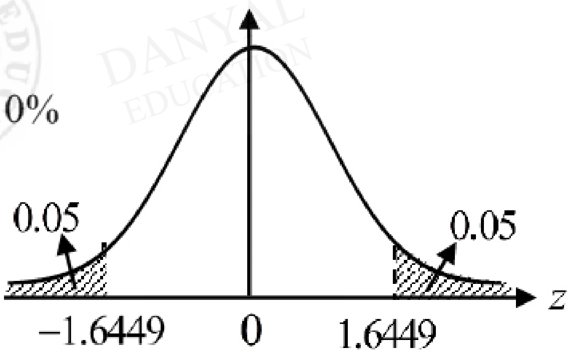
iv There is a probability of 0.10 that the test will conclude the population mean waiting time is more than 240 seconds when it is actually 240 seconds.

v $H_0 : \mu = k$

$H_1 : \mu \neq k$

Level of significance: 10%

For H_0 to be rejected,



$$z \le -1.6449 \text{ or } z \ge 1.6449$$

$$\frac{\bar{x}-k}{s/\sqrt{50}} \le -1.6449 \text{ or } \frac{\bar{x}-k}{s/\sqrt{50}} \ge 1.6449$$

$$\frac{242.4-k}{14.923/\sqrt{50}} \le -1.6449 \text{ or } \frac{242.4-k}{14.923/\sqrt{50}} \ge 1.6449$$

$$242.4-k \le -3.4714 \text{ or } 242.4-k \ge 3.4714$$

$$k \ge 245.87 \text{ or } k \le 238.93$$

$$\{k \in \mathbb{R} : k \le 239 \text{ (3 s.f) or } k \ge 246 \text{ (3 s.f)}\}$$

Q3

(i)

To test

$H_0 : \mu = 0.5$

$H_1 : \mu \neq 0.5$

> Two-tailed test as the question asked for the same diameter **or not.**

Level of significance: 5%

Under H_0, $Z = \dfrac{\bar{X} - 0.5}{0.04/\sqrt{25}} \sim N(0,1)$

> Under H_0, we have assumed that $\mu = 0.5$. Thus we have to replace μ by 0.5.
> Note that the population variance is given for this part!

Reject H_0 if p-value ≤ 0.05

Calculation:

$\bar{x} = 0.51$, p-value $= 0.211$

Since p-value > 0.05, we do not reject H_0. Thus there is insufficient evidence at 5% level of significance that the manufacturing process is producing ball bearings of different diameters.

Distribution of the diameter of the ball bearings is normal.

> As sample size is small.

On the whole, this question was badly done. Many candidates still showed poor presentation. They have also illustrated poor understanding in the writing of rejection region/criteria. Common mistakes include swapping μ_0 and $\bar{x}$; identify H_1 wrongly.

(ii)

To test

$H_0 : \mu = 0.5$

$H_1 : \mu > 0.5$

Level of significance: 5%

Under H_0, by Central Limit Theorem, $Z = \dfrac{\bar{X} - 0.5}{s/\sqrt{100}} \sim N(0,1)$ approx

Reject H_0 if p-value ≤ 0.05

Calculation:

$\bar{x} = 0.506$, $\Sigma(x - 0.5) = 50.6 - 50 = 0.6$

> We must obtain $\Sigma(x-0.5)$ first before applying formula

$$s^2 = \frac{1}{n-1}\left[\Sigma(x-0.5)^2 - \frac{(\Sigma(x-0.5))^2}{n}\right]$$

$$= \frac{1}{99}\left(0.08345 - \frac{0.6^2}{100}\right)$$

$$= 8.07 \times 10^{-4}$$

> Note that $\Sigma(x-0.5)^2$ is not sample variance as 0.5 is not the sample mean!

p-value $= 0.0173$

Since p-value < 0.05, we reject H_0. Thus there is sufficient evidence at 5% level of significance that the manufacturing process is producing oversized ball bearings.

A handful of students identify this as a left tail test, resulting in a p-value that is more than 0.5. Note that for hypothesis testing, it does not make sense to have a p-value that is more than 0.5. As a rule of thumb, if $\bar{x} > \mu_0$, we should test $\mu > \mu_0$.

(iii)

For the new sample, $\bar{x} = 0.506$.

However,

$$s^2 = \frac{1}{199}\left(2(0.08345) - \frac{(2(0.6))^2}{200}\right)$$

$$= 8.02513 \times 10^{-4}$$

> Combined sample is **NOT** pooled sample. Please do not use this formula at all! (It's for Further Math)

$Z_{calc} = 2.995$, p-value $= 0.00137$

Since p-value < 0.05, we will still reject H_0.

The conclusion remains the same.

For this part, candidates are supposed to either compute the p-value after the samples are combined, or give clearly supported reason why the p-value will be smaller.