A Level H2 Math

Graphs and Transformations Test 4

Q1

A curve C_1 has equation $y=\dfrac{ax^2-bx}{x^2-c}$, where a, b and c are constants. It is given that C_1 passes through the point $\left(3, \frac{9}{5}\right)$ and two of its asymptotes are $y=2$ and $x=-2$.

(i) Find the values of a, b and c. [3]

In the rest of the question, take the values of a, b and c as found in part **(i)**.

(ii) Using an algebraic method, find the exact set of values of y that C_1 cannot take. [3]

(iii) Sketch C_1, showing clearly the equations of asymptotes and the coordinates of the turning points. [3]

(iv) It is given that the equation $e^y = x-r$, where $r \in \mathbb{Z}^+$, has exactly one real root. State the range of values of r. [1]

(v) The curve C_2 has equation $y=2+\dfrac{3x+5}{x^2-2x-3}$. State a sequence of transformations which transforms C_1 to C_2. [3]

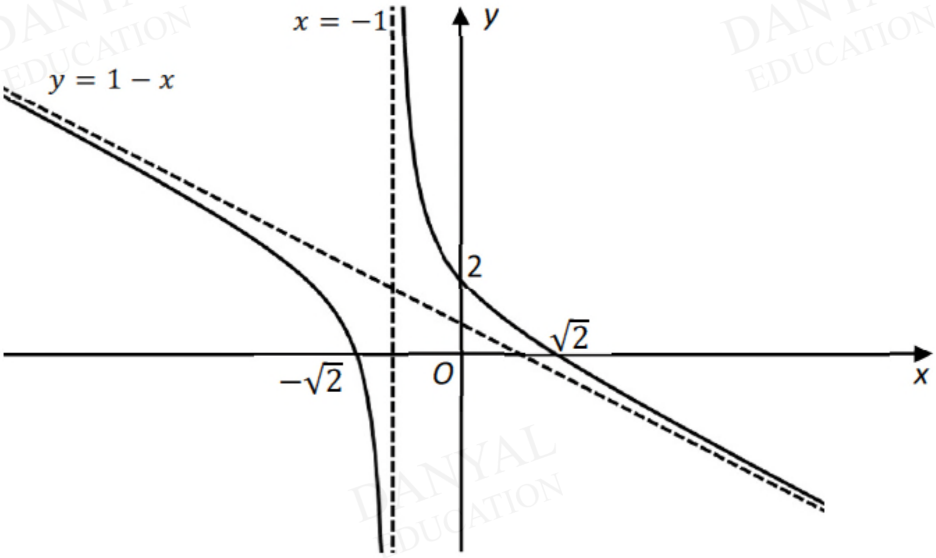
Q2

(a) The curve C has the equation

$$(x-2)^2 = a^2\left(1-y^2\right), \qquad 1<a<2.$$

Sketch C, showing clearly any intercepts and key features. [2]

(b) The diagram shows the graph of $y = \mathrm{f}(x)$, which has an oblique asymptote $y = 1-x$, a vertical asymptote $x=-1$, x-intercepts at $\left(\sqrt{2},0\right)$ and $\left(-\sqrt{2},0\right)$, and y-intercept at $(0,2)$.



Sketch, on separate diagrams, the graphs of

(i) $y = \dfrac{1}{\mathrm{f}(x)}$, [3]

(ii) $y = \mathrm{f}'(x)$, [3]

showing clearly all relevant asymptotes and intercepts, where possible.

Q3

The curve C has equation $y=\dfrac{4x^2-kx+2}{x-2}$, where k is a constant.

(i) Show that curve C has stationary points when $k<9$. [3]

(ii) Sketch the graph of C for the case where $6<k<9$, clearly indicating any asymptotes and points of intersection with the axes. [4]

(iii) Describe a sequence of transformations which transforms the graph of $y=2x+\dfrac{1}{x}$ to the graph of $y=\dfrac{4x^2-8x+2}{x-2}$. [3]

(iv) By drawing a suitable graph on the same diagram as the graph of C, solve the inequality

$$\frac{4x^2-8x+2}{x-2}>\frac{1}{x^2}.$$ [3]

Answers

Graphs and Transformations Test 4

Q1

(i)

$$y = \frac{ax^2 - bx}{x^2 - c}$$

Since $y = 2$ is a horizontal asymptote, $a = 2$.
Since $x = -2$ is a vertical asymptote, $c = 4$.

$\left(3, \frac{9}{5}\right)$ lies on $y = \frac{2x^2 - bx}{x^2 - 4}$

$$\therefore \ \frac{9}{5} = \frac{2(3)^2 - b(3)}{(3)^2 - 4} \quad \Rightarrow \quad b = 3$$

(ii)

$$y = \frac{2x^2 - 3x}{x^2 - 4}$$

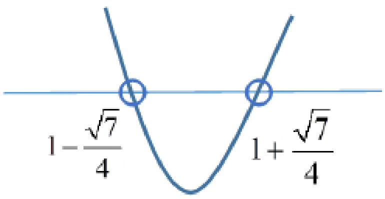
$$y\left(x^2 - 4\right) = 2x^2 - 3x$$

$$(y-2)x^2 + 3x - 4y = 0$$

For no real roots,

$$(3)^2 - 4(y-2)(-4y) < 0$$

$$16y^2 - 32y + 9 < 0$$



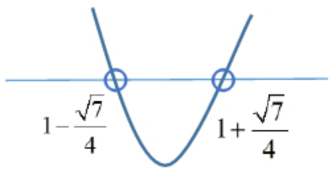
Method 1

$$\therefore\ y=\frac{32\pm\sqrt{(32)^2-4(16)(9)}}{2(16)}=\frac{32\pm\sqrt{448}}{32}=1\pm\frac{\sqrt{7}}{4}$$

$\therefore$ required set is $\left\{y\in\mathbb{R}:1-\frac{\sqrt{7}}{4}<y<1+\frac{\sqrt{7}}{4}\right\}$.

Method 2 (completing the square)

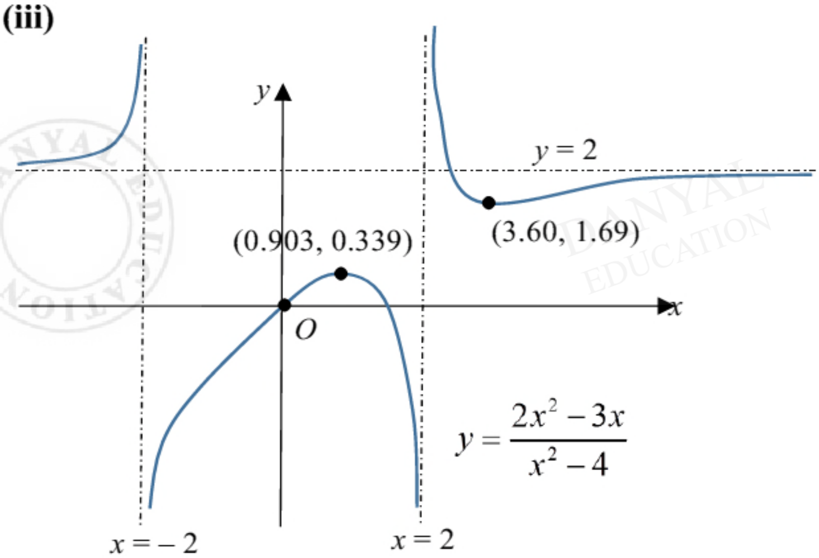
$$16y^2-32y+9<0$$
$$y^2-2y+\tfrac{9}{16}<0$$
$$(y-1)^2-\tfrac{7}{16}<0$$
$$\left(y-1+\tfrac{\sqrt{7}}{4}\right)\left(y-1-\tfrac{\sqrt{7}}{4}\right)<0$$



$\therefore\ 1-\frac{\sqrt{7}}{4}<y<1+\frac{\sqrt{7}}{4}$

$\therefore$ required set is $\left\{y\in\mathbb{R}:1-\frac{\sqrt{7}}{4}<y<1+\frac{\sqrt{7}}{4}\right\}$

(iii)

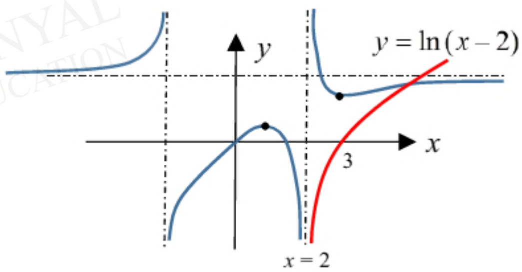


(iv)

$e^y=x-r$

$y=\ln(x-r)$

$r\geq 2$



(v)

$$C_1 : y = \frac{2x^2 - 3x}{x^2 - 4} = 2 + \frac{8 - 3x}{x^2 - 4}$$

$$\begin{array}{r} 2 \\ x^2 - 4 \overline{\smash{)}\, 2x^2 - 3x} \\ \underline{2x^2 - 8} \\ -3x + 8 \end{array}$$

$$C_2 : y = 2 + \frac{3x + 5}{x^2 - 2x - 3}$$

$$= 2 + \frac{3x + 5}{(x-1)^2 - 4}$$

$$= 2 + \frac{8 - 3(1 - x)}{(1 - x)^2 - 4}$$

<u>Method 1</u>

Transformation: $x \to x+1 \to -x+1$

1. Translation of C_1 1 unit in the negative x-direction to get

$y = 2 + \dfrac{8 - 3(x+1)}{(x+1)^2 - 4} = 2 + \dfrac{-3x+5}{x^2 + 2x - 3}$ followed by

2. Reflection of $y = 2 + \dfrac{-3x+5}{x^2 + 2x - 3}$ in the y-axis to get C_2.

<u>Method 2</u>

Transformation: $x \to -x \to -(x-1) = -x+1$

1. Reflection of C_1 in the y-axis to get $y = 2 + \dfrac{8+3x}{x^2 - 4}$

followed by

2. Translation of $y = 2 + \dfrac{8+3x}{x^2 - 4}$ 1 unit in the positive x-direction to get C_2.

Q2

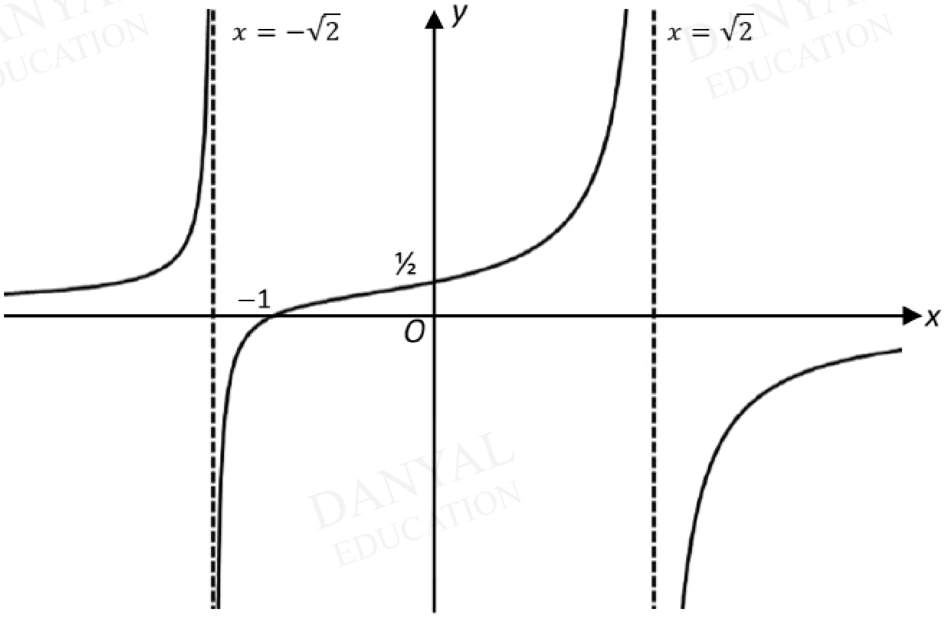
(a) $(x-2)^2 = a^2(1-y^2)$

$\Rightarrow \dfrac{(x-2)^2}{a^2} + y^2 = 1$

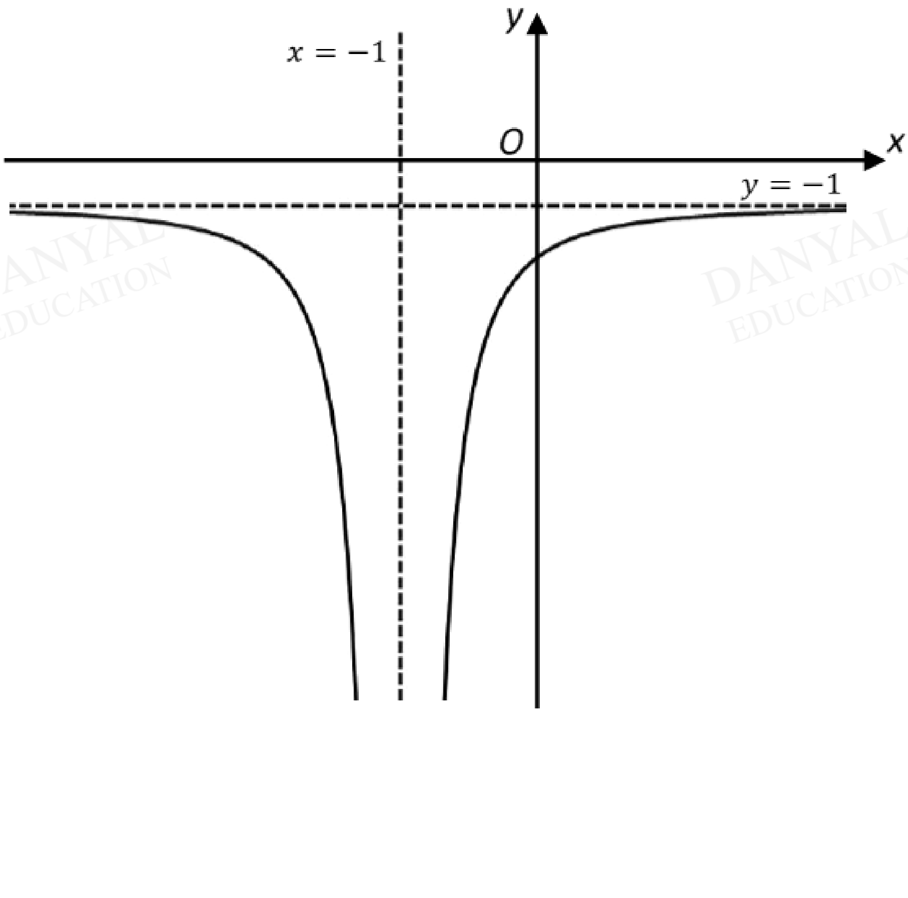
$\Rightarrow \dfrac{(x-2)^2}{a^2} + \dfrac{(y-0)^2}{1^2} = 1,$

$1 < a < 2$

(b)(i) $y = \dfrac{1}{\mathrm{f}(x)}$



(b)(ii) $y = \mathrm{f}'(x)$



Q3

i)

$$y=\frac{4x^2-kx+2}{x-2}$$

By long division, $y=4x+8-k+\frac{18-2k}{x-2}$

$$\frac{dy}{dx}=\frac{(x-2)(8x-k)-(4x^2-kx+2)(1)}{(x-2)^2}$$

$$=\frac{4x^2-16x+2k-2}{(x-2)^2}$$

Let $\frac{dy}{dx}=0 \Rightarrow 4x^2-16x+2k-2=0$

$$\Rightarrow 2x^2-8x+k-1=0$$

$$\Rightarrow x=\frac{8\pm\sqrt{64-4(2)(k-1)}}{4}=2\pm\sqrt{\frac{9-k}{2}}$$

C has stationary point when $k\le 9$
However, when $k=9$, the value x=2 is undefined on the curve.
In fact, the curve C is a straight line, $y = 4x - 1$.
Hence C has stationary point when $k < 9$.

Alternative Presentation 1:

Let $\frac{dy}{dx}=0 \Rightarrow 4x^2-16x+2k-2=0$

$$\Rightarrow 2x^2-8x+k-1=0$$

For $\frac{dy}{dx}$=0 to have real roots, "$b^2-4ac\ge 0$"

$$\Rightarrow 8^2-4(2)(k-1)\ge 0$$

$$\Rightarrow 64-8k+8\ge 0$$

$$\Rightarrow 8k\le 72$$

$$\Rightarrow k\le 9$$

Alternative Presentation 2:

$\frac{dy}{dx}=0 \Rightarrow 4x^2-16x+2k-2=0$

$$\Rightarrow 2x^2-8x+k-1=0$$

$$\Rightarrow 2(x-2)^2+k-9=0$$

$$\Rightarrow 2(x-2)^2=9-k$$

For $\frac{dy}{dx}=0$ to have roots x,

$9-k\ge 0 \Rightarrow k\le 9$

However, when $k=9$, the value x=2 is undefined on the curve.
In fact, the curve C is a straight line, $y = 4x - 1$.
Hence C has stationary point when $k < 9$.

(ii)

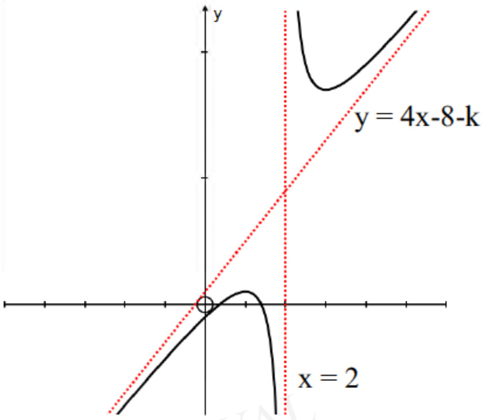
$$y=\frac{4x^2-kx+2}{x-2}=4x+(8-k)+\frac{18-2k}{x-2}$$

Asymptotes of C are $y = 4x + 8 - k$ and $x = 2$
When $x=0$, $y=-1$.
When $y=0$, $4x^2-kx+2=0$

$$\Rightarrow x=\frac{k\pm\sqrt{k^2-32}}{8}$$

The axial intercepts are $(0,-1)$, $\left(\frac{k-\sqrt{k^2-32}}{8},0\right)$ and $\left(\frac{k+\sqrt{k^2-32}}{8},0\right)$.

ii)



(iii)

When k = 8, $y=4x+(8-8)+\frac{18-2(8)}{x-2}=4x+\frac{2}{x-2}$

$y=2x+\frac{1}{x}\xrightarrow{A}y=2\left(2x+\frac{1}{x}\right)=4x+\frac{2}{x}$

$y=4x+\frac{2}{x}\xrightarrow{B}y=4(x-2)+\frac{2}{(x-2)}=y=4x-8+\frac{2}{(x-2)}$

$y=4x-8+\frac{2}{(x-2)}\xrightarrow{C}y=\left(4x-8+\frac{2}{(x-2)}\right)+8=4x+\frac{2}{(x-2)}$

A – Translate the graph by 2 units in the direction of x-axis
B - Scaling, parallel to the y-axis by a scale factor of 2.
C - Translate the graph by 8 units in the direction of y-axis

Alternate Sequence of Transformations:
A – Translate the graph by 2 units in the direction of x-axis
B - Translate the graph by 4 units in the direction of y-axis
C - Scaling, parallel to the y-axis by a scale factor of 2.

iv)

When $k=8$, $\frac{4x^2-kx+2}{x-2}>\frac{1}{x^2}$

$\Rightarrow \frac{4x^2-8x+2}{x-2}>\frac{1}{x^2}$

From G.C,
$0.805<x<1.69$ or $x>2$.