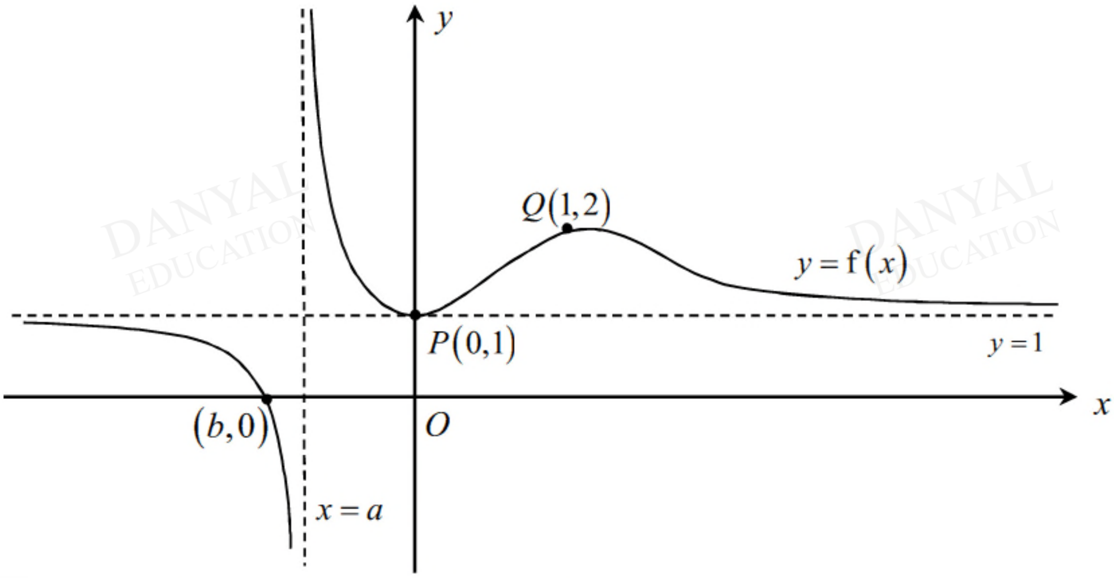
A Level H2 Math

Graphs and Transformations Test 1

Q1

The diagram below shows the graph of $y=\mathrm{f}(x)$.



The graph passes through the point $(b,0)$ and has turning points at $P(0,1)$ and $Q(1,2)$. The lines $y=1$ and $x=a$, where $b<a<-\frac{1}{2}$, are asymptotes to the curve.

On separate diagrams, sketch the graphs of

(i) $y=\mathrm{f}\left(\frac{x-1}{2}\right)$, [3]

(ii) $y=\mathrm{f}'(x)$, [3]

labelling, in terms of a and b where applicable, the exact coordinates of the points corresponding to P and Q, and the equations of any asymptotes.

Q2

The graph of $y = \dfrac{x-1}{ax^2+bx+c}$, where a, b and c are non-zero constants, has a turning point at $(-1,1)$, and an asymptote with equation $x = -\dfrac{1}{3}$. Find the values of a, b and c. [5]

Q3

The function p is defined by $\mathrm{p}: x \mapsto \dfrac{1-x^2}{1+x^2}$, $x \in \mathbb{R}$.

(i) Find algebraically the range of p, showing your working clearly. [3]

(ii) Show that $\mathrm{p}(x) = \mathrm{p}(-x)$ for all $x \in \mathbb{R}$. [1]

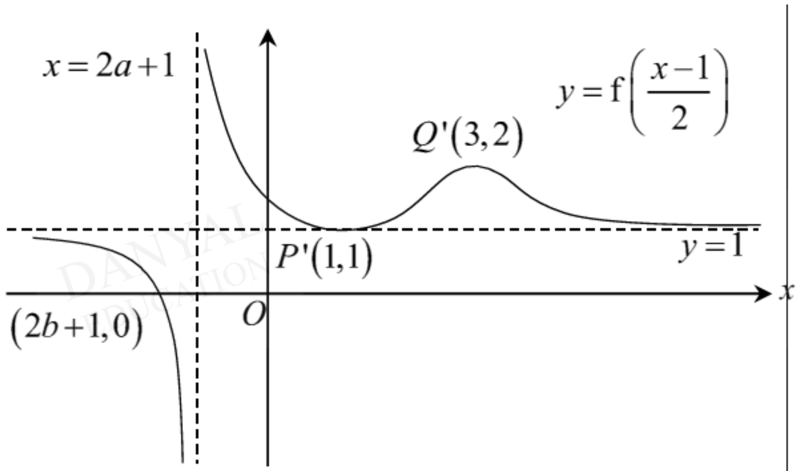
It is given that $\mathrm{q}(x) = \mathrm{p}\left(\dfrac{1}{2}x - 4\right)$, $x \in \mathbb{R}$.

(iii) State a sequence of transformations that will transform the graph of p on to the graph of q. Hence state the line of symmetry for the graph of q. [3]

Answers

Graphs and Transformations Test 1

Q1

<table>
<tr>
<td>(i)</td>
<td>
</td>
<td>Almost the whole cohort gets either full marks or 1 mark (shape of the curve) for this question. Students has difficulty in handling $\frac{x-1}{2}$. Most students failed to read it as $\frac{x}{2}-\frac{1}{2}$. Thus the common mistake majority did was a translation of 1 unit in the positive $x-$direction followed by a stretching of factor 2 parallel to the $x-$axis.</td>
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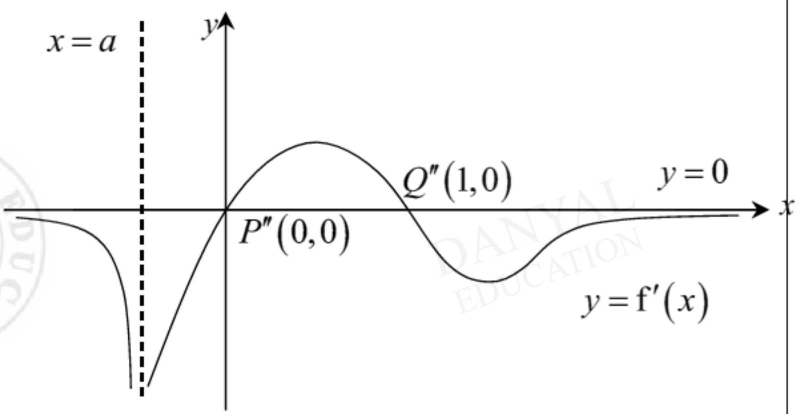
<td>(ii)</td>
<td>
</td>
<td>About 80% of the students are able to identify the asymptotes $x=a$, $y=0$ and the x-axis intercepts $P''(0,0)$, $Q''(1,0)$. Of these students, about 70% got full marks as some students couldn’t get the shape of the curve. Most students remembered to write the points in coordinates.</td>
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Q2

Passes through $(-1,1)$:

$$1=\frac{-2}{a-b+c}\ \Rightarrow\ a-b+c=-2\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Turning point at $(-1,1)$:

$$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{x=-1}=0$$

now $$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\left(ax^2+bx+c\right)-(x-1)(2ax+b)}{\left(ax^2+bx+c\right)^2}$$

Hence

$$\frac{(a-b+c)-(-2)(-2a+b)}{(a-b+c)^2}=0$$

$$\Rightarrow\ (a-b+c)-(-2)(-2a+b)=0$$

$$\Rightarrow\ -3a+b+c=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

When $x=-\frac{1}{3}$, $ax^2+bx+c=0$:

Hence $$\frac{a}{9}-\frac{b}{3}+c=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

Solving (1), (2) and (3) simultaneously, we get $a=3$, $b=7$ and $c=2$.

y

Some students forgot that the turning point (-1,1) lies on the curve and failed to substitute the point into the given equation to get an essential equation required for solving the unknowns.

Some students made mistakes when differentiating using the product or quotient rule, or incorrectly rewrote y as $y=(x-1)(ax^2+bx+c)$ instead of $y=(x-1)(ax^2+bx+c)^{-1}$ which also resulted in an incorrect derivative.

Some students did not know how to handle the information given on the asymptote. Some completed the square or did long division (both not necessary) and came up with an incorrect equation/conclusion.

Some wrongly assumed that since $x=-\frac{1}{3}$ is an asymptote, therefore,

$$\rightarrow ax^2+bx+c=\left(x+\frac{1}{3}\right)(x-c)$$

$$\rightarrow ax^2+bx+c=(3x+1)(x-c)$$

$$\rightarrow ax^2+bx+c=\left(x+\frac{1}{3}\right)^2$$

$$\rightarrow ax^2+bx+c=(3x+1)^2$$

which made assumptions on the values of *a*; those who assumed *a*=3 might have obtained the same final answer because *a* happened to be 3 in this case, but the method was incorrect.

Q3

Let $y=\dfrac{1-x^2}{1+x^2},\ x\in\mathbb{R}$:

$y(1+x^2)=1-x^2$

$(y+1)x^2+(y-1)=0$

Discriminant ≥ 0: $\quad 0^2-4(y+1)(y-1)\geq 0$

$$-4(y^2-1)\geq 0$$

$$y^2-1\leq 0$$

$$y^2\leq 1$$

$$-1\leq y\leq 1$$

Since $y=-1$ is an asymptote, $\quad -1<y\leq 1$

<u>Alternative Method:</u>

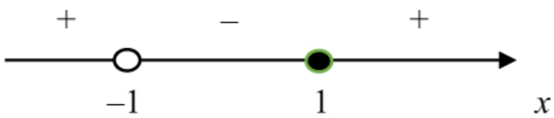
Let $y=\dfrac{1-x^2}{1+x^2},\ x\in\mathbb{R}$:

$y(1+x^2)=1-x^2$

$(y+1)x^2+(y-1)=0$

$x^2=\dfrac{1-y}{y+1},\ y\neq -1$

Since $x^2\geq 0\ \forall x\in\mathbb{R}$, $\quad \dfrac{1-y}{y+1}\geq 0$



$\therefore -1<y\leq 1$

(ii)

$$\mathrm{p}(-x)=\frac{1-(-x)^2}{1+(-x)^2}$$

$$=\frac{1-x^2}{1+x^2}$$

$=\mathrm{p}(x)\quad$ for all $x\in\mathbb{R}$ (shown)

(iii)

Graph of $\mathrm{q}(x)=\mathrm{p}\left(\dfrac{1}{2}x-4\right),\ x\in\mathbb{R}$ is obtained from the graph of $\mathrm{p}(x)$ by:

- Translation by 4 units in the positive x-direction, followed by

Stretch of factor 2 parallel to the x-axis.