A Level H2 Math

Functions Test 7

Q1

The function f is defined by

$$f : x \mapsto \ln(x^2 - 1),\ x \in \mathbb{R},\ x > 1.$$

(i) Find f^{-1} in similar form. [3]

(ii) Sketch f, f^{-1} and $f^{-1}f$ on the same diagram, indicating clearly all asymptotes and axial intercepts. [3]

The functions g and h are defined by

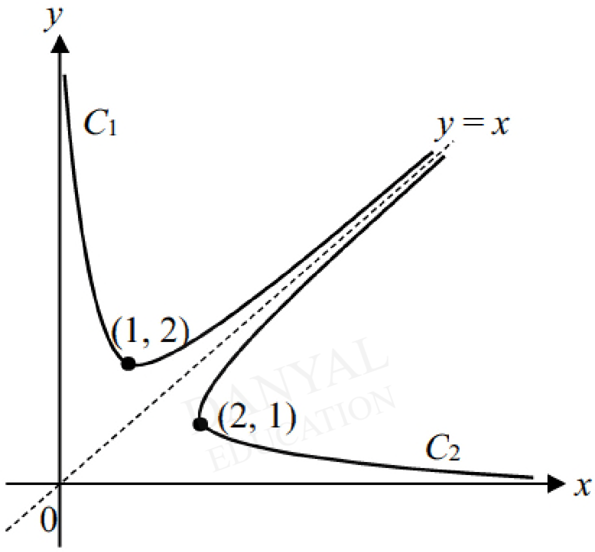
$$g : x \mapsto \begin{cases} 4(x-1)^2 & \text{for } 0 \le x < 2, \\ 8 - |2x - 8| & \text{for } 2 \le x < 8, \end{cases}$$

$$h : x \mapsto 3\sin x, \qquad 0 \le x \le \pi.$$

(iii) Sketch the graph of $y = g(x)$. [3]

(iv) Prove that the function gh exists and find the range of gh. [2]

Q2



(a) The diagram above shows two curves C_1 and C_2 which are reflections of each other about the line $y = x$. State with justification, whether the following statement is true: “If C_1 is the graph of $y = \mathrm{f}(x)$, then C_2 is the graph of $y = \mathrm{f}^{-1}(x)$.”

(b) The functions f and g are defined as follows

$$\mathrm{f}: x \mapsto \frac{1}{x^2 - x - 6}, \quad x \in \mathbb{R}, x < 0, \ x \neq -2$$

$$\mathrm{g}: x \mapsto \tan^{-1}\left(\frac{x}{2}\right), \quad x \in \mathbb{R}$$

(i) Sketch the graph of $y = \mathrm{f}(x)$. Determine whether f^2 exists. [3]

(ii) Find $\mathrm{f}^{-1}(x)$. [2]

(iii) Given that $\mathrm{gf}(a) = \dfrac{\pi}{4}$, find the exact value of a. [2]

Q3

Functions g and h are defined by

$$\mathrm{g}: x \mapsto x^2 + 6x + 8, \ x \in \mathbb{R}, \ x \leq \alpha,$$

$$\mathrm{h}: x \mapsto -\mathrm{e}^x, \ x \in \mathbb{R}, \ x > -2.$$

(i) Given that the function g^{-1} exists, write down the largest value of α and define g^{-1} in similar form. State a transformation which will transform the curve $y = \mathrm{g}(x)$ onto the curve $y = \mathrm{g}^{-1}(x)$. [5]

(ii) Given instead that $\alpha = -2$, explain why the composite function hg exists and find the exact range of hg. [2]

Answers

Functions Test 7

Q1

(i)

Let $y = \ln\left(x^2 - 1\right)$.

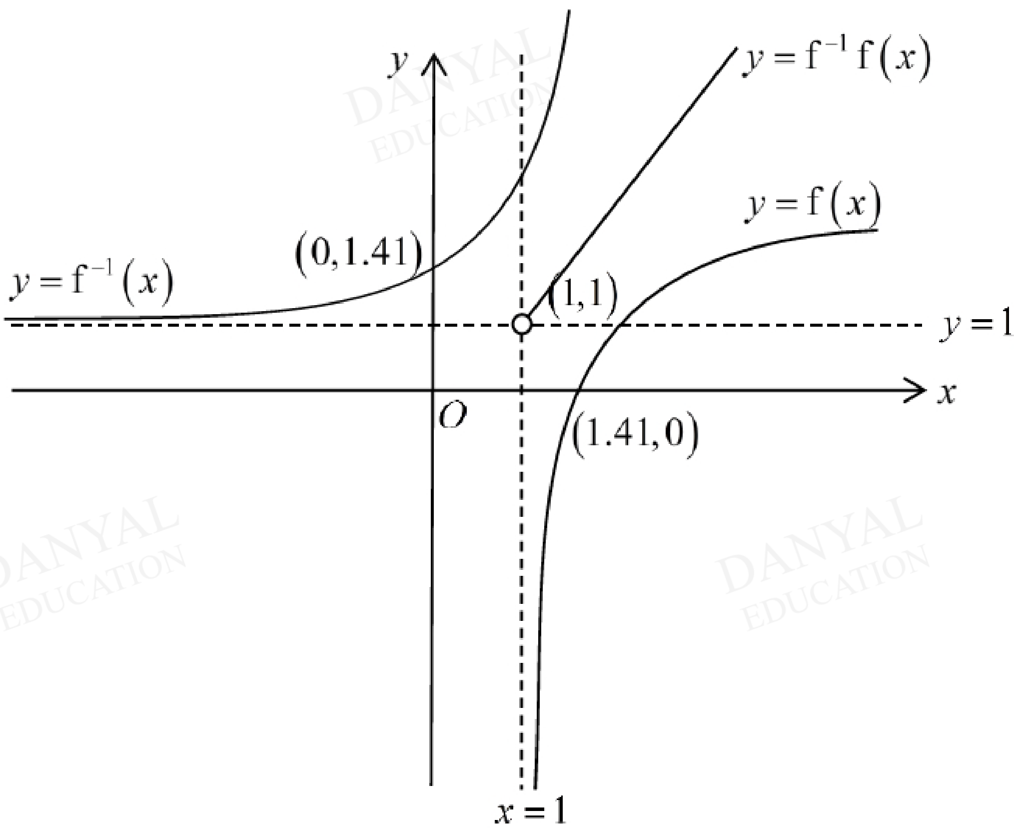
$x = \pm\sqrt{1+e^y}$

Since $x > 1 > 0$, $\therefore x = \sqrt{1+e^y}$.

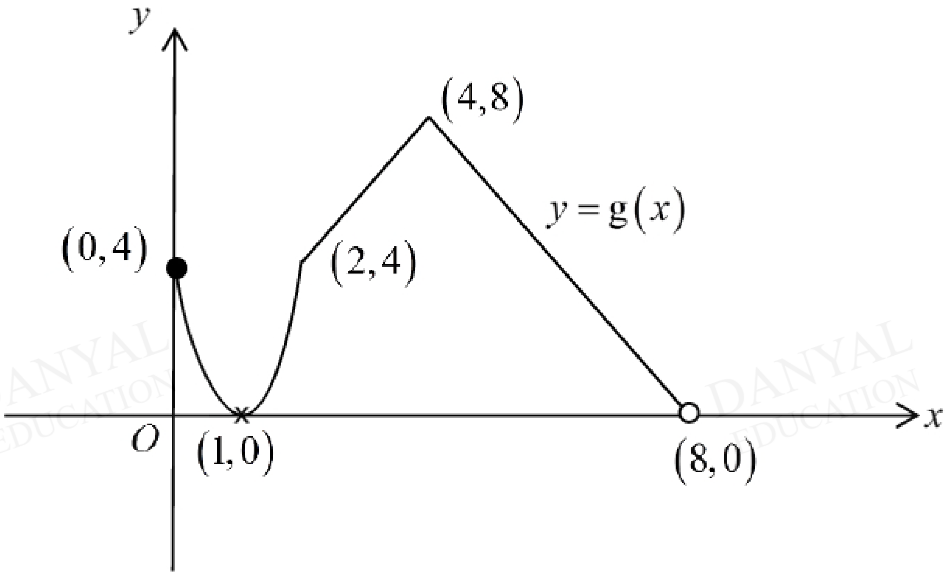
$$D_{f^{-1}} = R_f$$
$$= \mathbb{R}$$

$f^{-1} : x \mapsto \sqrt{1+e^x}$, $x \in \mathbb{R}$

(ii)



(iii)



(iv) Since $R_h = [0,3] \subseteq [0,8) = D_g$, therefore the function gh exists.

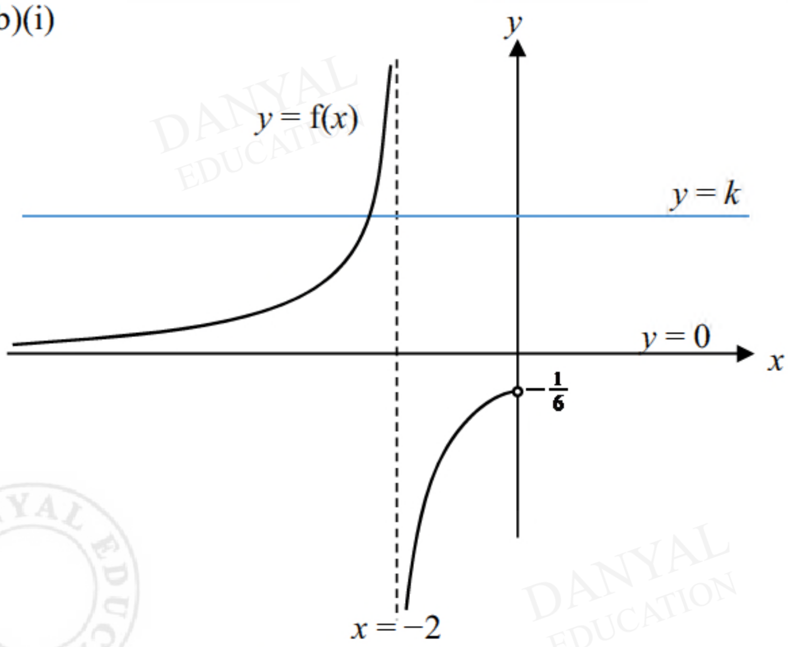
Restrict D_g to be $[0,3]$

From the graph in **(iii)**, $R_{gh} = [0,6]$.

Q2

(a) From the graph of $y = \mathrm{f}(x)$ which is C_1, there exists a horizontal line $y = 3$ which cuts the graph of $y = \mathrm{f}(x)$ at 2 points. f is not one to one and thus f^{-1} does not exist. Since f^{-1} does not exist, C_2 is not the graph of $\mathrm{f}^{-1}(x)$.	

(b)(i)



Since $R_\mathrm{f} = \left(-\infty, -\frac{1}{6}\right) \cup (0, \infty)$

and $D_\mathrm{f} = (-\infty, 0)$

i.e. $R_\mathrm{f} \not\subset D_\mathrm{f}$

Therefore, f^2 does not exist.

(b)(ii) Let $y = \dfrac{1}{x^2 - x - 6}$

$$\Rightarrow \quad yx^2 - xy - 6y - 1 = 0$$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{y^2 + 4y(6y+1)}}{2y}$$

$$\Rightarrow \quad x = \frac{y \pm \sqrt{25y^2 + 4y}}{2y}$$

Since $x < 0$, $x = \dfrac{y - \sqrt{25y^2 + 4y}}{2y} = \dfrac{1}{2} - \dfrac{\sqrt{25y^2 + 4y}}{2y}$

Thus, $\mathrm{f}^{-1}(x) = \dfrac{1}{2} - \dfrac{\sqrt{25x^2 + 4x}}{2x}$.

(b)(iii) $\text{gf}(a) = \dfrac{\pi}{4}$

$$\tan^{-1}\left(\frac{\text{f}(a)}{2}\right) = \frac{\pi}{4}$$

$$\frac{\text{f}(a)}{2} = 1$$

$$\text{f}(a) = 2 \quad \Rightarrow \quad a = \text{f}^{-1}(2)$$

$$\therefore a = \frac{1}{2} - \frac{\sqrt{25(2)^2 + 4(2)}}{2(2)} = \frac{1}{2} - \frac{3\sqrt{3}}{2}$$

Alternative solution

$$\text{gf}(a) = \frac{\pi}{4}$$

$$\tan^{-1}\left(\frac{\text{f}(a)}{2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \quad \frac{\text{f}(a)}{2} = 1$$

$$\Rightarrow \quad \text{f}(a) = 2$$

$$\Rightarrow \quad \frac{1}{a^2 - a - 6} = 2$$

$$\Rightarrow \quad 2a^2 - 2a - 13 = 0$$

$$\Rightarrow \quad a = \frac{2 \pm \sqrt{4 + 4(2)(13)}}{4} = \frac{2 \pm 6\sqrt{3}}{4} = \frac{1}{2} \pm \frac{3}{2}\sqrt{3}$$

Since $a < 0$, $a = \dfrac{1}{2} - \dfrac{3}{2}\sqrt{3}$

Marker's comments

(a) This part is generally well-answered by students who recognised that f is not 1-1 and so the inverse cannot exist. Students who gave no or incorrect justification to why the statement is false fail to gain any credit.

(b) The graph of $y = \text{f}(x)$ is generally well-drawn and most students were able to present their sketches within the correct domain. The main issue for this part is that many students tried to justify whether f^2 exists or not in relation to whether the inverse function exists or not, showing a misconception between composite and inverse functions. Many students proceeded to obtain full credits for the desired results of parts (ii) and (iii), but students need to first be aware that no credit was deducted when the domain or range was presented wrongly.

Q3

(i) Largest $\alpha = -3$

Let $y = \mathrm{g}(x) = x^2 + 6x + 8$
$= (x+3)^2 - 1$

$(x+3)^2 = y + 1$
$x + 3 = \pm\sqrt{y+1}$

$x = -3 \pm\sqrt{y+1}$
Since $x \le -3$, $x = -3 - \sqrt{y+1}$

$\mathrm{g}^{-1} : x \mapsto -3 - \sqrt{x+1}, \ x \in [-1, \infty)$

A reflection about the line $y = x$ will transform the curve $y = \mathrm{g}(x)$ onto the curve $y = \mathrm{g}^{-1}(x)$.

(ii) Since $\mathrm{R_g} = [-1, \infty) \subseteq (-2, \infty) = \mathrm{D_h}$, the composite function hg exists.

$\mathrm{R_{hg}} = \left(-\infty, -\frac{1}{\mathrm{e}}\right]$